Problem 1

The detection metric is given as:

\[ d^2 = \left( \frac{E(L | H_1) - E(L | H_0)}{\sigma^2_{L | H_0}} \right)^2 \]

\[ E(L | H_1) = E[ ||s + w||^2 ] = E[ (s^2 + w^2) (s + w) ] = \]

\[ = E[ s^2 s + s^2 w + w^2 s + w^2 w ] = E[ ||s||^2 ] + E[ s^2 ] E[ w^2 ] + E[ w^2 ] E[ s^2 ] + E[ ||w||^2 ] \]

Since \( w \) and \( s \) are independent.

Then

\[ E[ L | H_1 ] = E[ ||s||^2 ] + E[ ||w||^2 ] \]

\[ = E[ ||s||^2 ] + \sum_{i=1}^{N} E[w_i^2] = \sum_{i=1}^{N} \lambda_i + N \sigma_w^2 \]

\[ E[ L | H_0 ] = E[ ||w||^2 ] = N \sigma_w^2 \]

\[ \sigma^2_{L | H_0} = E[ L^2 | H_0 ] - E[ L | H_0 ]^2 = E[ L^2 | H_0 ] - (N \sigma_w^2)^2 \]

\[ E[ L^2 | H_0 ] = E[ ||w||^4 ] = E[ ||w||^2 ]^2 = E[ (w^4)^2 ] = \]

\[ = E \left[ \sum_{i=1}^{N} w_i \sum_{i=1}^{N} w_i w_j \right] = E \left[ \sum_{i \neq j} w_i w_j \right] \]

If \( i = j \) then \( E[ L^2 | H_0 ] = \sum_{i=1}^{N} E[ w_i^4 ] = \sum_{i=1}^{N} E[ w_i^4 ] = \sum_{i=1}^{N} \lambda_i = \sum_{i=1}^{N} \lambda_i = N \sigma_w^4 \)

from the moment then for Gaussians we have

\[ E[ w_i w_j w_i w_j ] = 2 \sigma_w^4 \]

hence

\[ E[ L^2 | H_0 ] = 2 N \sigma_w^4 \]

If \( i \neq j \) then

\[ E[ L^2 | H_0 ] = \sum_{i \neq j} E[ w_i^2 ] E[ w_j^2 ] = (N^2 - N) \sigma_w^4 \]
Hence, \( E[\mathcal{L}^2|\mathcal{H}_0] = (N^2 + N)\sigma_w^4 \)

and \( \sigma^2_{\mathcal{L}|\mathcal{H}_0} = E[\mathcal{L}^2|\mathcal{H}_0] - E[\mathcal{L}|\mathcal{H}_0] = N\sigma_w^4 \)

So the detection metric is

\[
d^2 = \left( \frac{\sum_{i=1}^{N} x_i + N\sigma_w^2 - N\sigma_w^2}{N\sigma_w^4} \right)^2 = \left( \frac{\sum_{i=1}^{N} x_i}{N\sigma_w^2} \right)^2
\]

\( \Rightarrow \)

\( d^2 = \frac{E_s^2}{N\sigma_w^4} \)

which is a signal-to-noise ratio.

\( \text{ii}) \)

if we have a power constraint (this happens always in practice) then

\( d^2 \) obviously to maximize \( d^2 \) we better choose \( N=1 \). This means it is better to put all the power to one signal \( (s_i) \) than to distribute the available power among different signals because the resulting SNR per signal \( \left( \frac{\lambda_i^2}{\sigma_w^4} \right) \) is higher when \( \lambda_i = E_s \).
Problem 2

\[ E[y(t)] = E[s(t)a(t) + (1 - s(t))b(t)] = E(s(t))E[a(t)] + (1 - E[s(t)])E[b(t)] \]

We know that \( R_a(z) = Pa \cos(z \pi a z) e^{-12z/2} \)

\[ = E[a(t+z) a^*(t)] \]

Note that \( \lim_{z \to \infty} E[a(t+z) a^*(t)] = E[a(t)]E[a^*(t)] \)

\( a(t) \) is WSS so \( E[a(t)] \) is constant.

\[ |m_a|^2 = \lim_{z \to \infty} |Pa \cos(z \pi a z) e^{-12z/2}| = \lim_{z \to \infty} Pa e^{-12z/2} \]

Thus \( |m_a|^2 = \lim_{z \to \infty} Pa \cos(z \pi a z) e^{-12z/2} \)

\( = 0 \) hence \( m_a = 0 \)

Similarly, \( m_b = 0 \).

Hence, \( E[y(t)] = 0 \).

\[ E[y(t+z)y^*(t)] = E[(s(t+z)a(t+z) + (1-s(t+z))b(t+z))(s(t)a^*(t) + (1-s(t))b^*(t))] \]

Since all signals are uncorrelated we get

\[ E[y(t+z)y^*(t)] = R_s(z)R_a(z) + E[s(t+z)]E[a^*(t)] + (-R_s(z) + E[s^*(t)])E[a(t)]E[b(t+z)] + \]

\[ + R_s(z)[E[a(t)]E[b(t+z)] + R_s(z)] - E[s^*(t)]E[s(t+z)] + R_s(z) \]

\[ = R_s(z)R_a(z) + (m_s - R_s(z))m_a m_b^* + (R_s(z) + m_s^*)m_a^* m_b + R_b(z)(1 - m_s - m_s^* + R_s(z)) \]

\[ |m_a|^2 = \lim_{z \to \infty} \frac{1}{4} (e^{-12z/2} + 1) = \frac{1}{4} \quad \Rightarrow \quad m_s = \frac{1}{2} \quad (s(t) \text{ is real}) \]
Thus $E[y(t+\tau)y^*(t)] = R_y(\tau) = \frac{1}{12}(e^{-\lambda\frac{1}{2}t^2} + 2) (\cos(e^{\eta_0} t) + \cos(e^{\eta_2} t)) \frac{R_y}{2} e^{-\lambda\frac{1}{2}t}$

Obviously $y(t)$ is WSS.

Recall that $e^{-\lambda\frac{1}{2}t^2} \leq \int_{-\infty}^{\infty} \frac{2\lambda}{(2\pi)^{1/2} a^2} e^{-\lambda t^2/(2a^2)} dt$

$R_y(\tau) = \frac{P_\alpha}{4} (e^{-\lambda\frac{1}{2}(\frac{t}{T} + 1)} + e^{-\lambda\frac{1}{2}T} + \cos(\eta_0 t) + \cos(\eta_2 t))$

$\Rightarrow S_y(f) = \frac{P_\alpha}{4} \left[ \frac{2(\lambda + \frac{1}{T})}{(2\pi^2 + (\lambda - \frac{1}{T})^2} + \frac{2\frac{1}{T}}{2\pi^2 + \frac{1}{T^2}} \right] \left( \delta(f - \eta_0) + \delta(f + \eta_0) \right) + \delta(f - \eta_2) + \delta(f + \eta_2)$

$= \frac{P_\alpha}{4} \left[ \frac{\lambda + \frac{1}{T}}{(2\pi^2 + (\lambda - \frac{1}{T})^2} + \frac{\frac{1}{T}}{2\pi^2 + \frac{1}{T^2}} \right] + \frac{\lambda + \frac{1}{T}}{(2\pi^2 + (\lambda + \frac{1}{T})^2} + \frac{\frac{1}{T}}{2\pi^2 + \frac{1}{T^2}} \right]$

The figure below shows the plot of $S_y(f)$ in dB

I used:

$f_b = 1000 \text{ Hz}$

$f_a = 100 \text{ Hz}$

$T = 1 \text{ sec}$

$\lambda = 10^{-2}$

A sample function of the ensemble would look like
Problem 3

\[ \sum_{k=0}^{M-1} y[n-k] = \sum_{k=0}^{M-1} x[n-k] \]

\[ b = \begin{bmatrix} 0.007 & 0.0000 & -0.1349 & 0.0000 & 0.0675 \end{bmatrix} \]
\[ a = \begin{bmatrix} 1.0000 & -1.0000 & 2.1192 & -1.2147 & 0.4128 \end{bmatrix} \]

(i) Plot \( H(e^{j\omega}) \)

See attached plot.

(ii) Plot the phase & group delays vs. frequency

See attached plot.

(iii) Implement the difference eq. when \( x[n] = f[n] \)

Max \( |y[n]| = 0.2409 \)

1% of \( y \) or response \( = 0.0024 \)

For \( n \geq 24 \) \( |y[n]| < 0.0024 \)

See attached plot of the difference equation response to a unit impulse input

(i.e. impulse response)

(iv) Determine the poles & zeros of the difference equation

Note: \[ H(e^{j\omega}) = \frac{b_1 e^{-j\omega} + b_2 e^{-j2\omega} + b_3 e^{-j3\omega} + b_4 e^{-j4\omega}}{a_1 + a_2 e^{-j\omega} + a_3 e^{-j2\omega} + a_4 e^{-j3\omega} + a_5 e^{-j4\omega}} \]

See the attached pole/zero plot.

Poles: \( 0.6540 \pm i0.5193 \)
\( 0.3172 \pm i0.7009 \)

Zeros: \( 0.9998 \pm i0.0192 \)
\( -0.9998 \pm i0.0192 \)
Problem 3 (i): $H(e^{j\omega})$
Problem 3 (ii): Phase delay

Normalized Frequency (\times \pi \text{ rad/sample})

Phase delay (samples)

Normalized Frequency (\times \pi \text{ rad/sample})

Group delay (samples)
Problem 3 (iii): $x[n]$: Filter input (unit impulse sequence)

$y[n]$: Difference equation response to a unit impulse sequence (impulse response)
Problem 3 (iv): Poles and zeros of the difference equation

- Poles
- Zeros
- Unit Circle
% Code for Problem Set 3, Problem 3

close all;
clear all;

b=[0.0675 0.0000 -0.1349 0.0000 0.0675];
a=[1.0000 -1.9425 2.1192 -1.2167 0.4128];

% i) Plot Frequency Response)
N=100000;
figure(1)
freqz(b,a,N)
title('Problem 3 (i): H(e^{j\omega})');

% ii) Plot the phase and group delays
figure(2)
subplot(211);
phasedelay(b,a);
title('Problem 3 (ii): Phase delay');
subplot(212);
grpdelay(b,a,N);
title('Group delay');

% iii) Plot the impulse response
n=-2:27
x=zeros(30,1);
x(3)=1;
y=filter(b,a,x);
figure(3);
subplot(211);
stem(n,x);
xlabel('n');
ylabel('x[n]');
title('Problem 3 (iii): x[n]: Filter input (unit impulse sequence)');
axis([-2 27, -.5 1.5]);
subplot(212);
stem(n,y);
xlabel('n');
ylabel('y[n]');
axis([-2 27, -.25 .25]);
title('y[n]: Difference equation response to a unit impulse sequence (impulse response)');

% iv) Find the poles and zeros
poles=roots(a);
zeros=roots(b);
figure(4);
theta=linspace(0,2*pi,1000);
plot(real(poles),imag(poles),'xr',real(zeros),imag(zeros),'ob',cos(theta),sin(theta),':ko');
legend('Poles','Zeros','Unit Circle');
axis([-1.1 1.1 -1.1 1.1]);
axis square;
xlabel('Real axis');
ylabel('Imaginary axis');
title('Problem 3 (iv): Poles and zeros of the difference equation');
Problem 4 - Transform to the discrete domain

\[ H(s) = \frac{s\omega_n}{s^2 + 2\xi\omega_n s + \omega_n^2} \quad \text{with resonant period } T_0 = \frac{2\pi}{\omega_n} \quad \text{and resonant frequency } \omega_n = \frac{1}{T_0} \]

i) Determine the corresponding difference equation when the derivative operations are

\[ \frac{dx(t)}{dt} \rightarrow x[n+1] - x[n-1] \quad \frac{\Delta T}{2} \]

\[ \frac{d^2 x(t)}{dt^2} \rightarrow x[n+1] - 2x[n] + x[n-1] \quad \frac{\Delta T}{2} \]

Note that the statement has an error in the denominator

\[ \frac{y[n]}{\Delta T^2} - \frac{3y[n]}{\Delta T} + y[n-1] + 2\omega_n^2 y[n+1] - y[n-1] = \omega_n^2 y[n] - y[n-1] \]

Corresponding difference equation:

\[ \left( \frac{1}{\Delta T} + \frac{s\omega_n}{2} \right)y[n+1] + \frac{1}{\Delta T}(\omega_n^2 \Delta T - \frac{2}{\Delta T})y[n] + \frac{1}{\Delta T}(-s\omega_n)y[n-1] = \frac{\omega_n}{2} \left( x[n-1] - x[n-2] \right) \]

ii) \[ H(z) = \frac{\omega_n}{2} \left( z - z^{-1} \right) \]

\[ \left( \frac{1}{\Delta T} + \frac{s\omega_n}{2} \right)z + \left( \omega_n^2 \Delta T - \frac{2}{\Delta T} \right) + \left( \frac{1}{\Delta T} - s\omega_n \right)z^{-1} \]
ii) Cont. See the attached plots of the impulse response for

\[ H(s) \] and the corresponding difference equations with

\[ DT = 0.001T_o, 0.01T_o, 0.1T_o \]. Also included are the

impulse responses for the bilinear mappings to the z-plane

for the same values of DT. (All plots for \( T_o = 4 \) yr, \( s = 0.25\))

Note that all of the impulse response functions have the
same general shape, but with different amplitudes

(which are a function of DT). For DT = 0.001T_o, 0.01T_o,

the impulse response dies out about as quickly as that

for H(s), but not quite as quickly for DT = 0.1T_o.

iii) See the attached pole-zero, DFT, and group-delay plots.

Note:

\[ \text{Group delay } H(s) = \frac{s^2 - \omega^2}{s^2 + \omega^2} \]

This expression is used in the following plots for group delay.

Bilinear transformation: Replace \( s \) in \( H(s) \) with \( \frac{2}{\pi DT} \).

After analyzing the attached plots of the impulse response,

the group delay, the pole/zero loci of the transfer function

DFTs, I conclude that the difference equation

approximation with \( DT = 0.001T_o \) comes closest to the fundamental

properties of the continuous system (peak response, decay rate, group delays, etc.)

(cont. on next page)
Both the bilinear $g$ difference eqn approximating with
$\Delta T = 0.01$ To seemed to do equally well when it
came to matching $H(s)$ near the resonant frequencies.
Also, the poles of zeros for those two were identical
(at least to any significant resolution). Most of the
impulse response functions appear to have identical
decay rates, though those with $\Delta T = 0.1$ To don't
decay quite as quickly as the others. The one
place where the diff eqn approx with $\Delta T = 0.001$ To
had a slight better match than the bilinear mapping
approx with $\Delta T = 0.001$ To is with the group delay
(see plot). For additional details, see notes on
attached plots.
Impulse response of $H(s)$ w/ $f_o=1$, $\delta=0.25$

Impulse Response w/ Diff. eqn. approx., $\Delta T=0.001*T_o$

Impulse Response w/ Bilinear Mapping, $\Delta T=0.001*T_o$

Impulse Response w/ Diff. eqn. approx., $\Delta T=0.01*T_o$

Impulse Response w/ Bilinear Mapping, $\Delta T=0.01*T_o$

Impulse Response w/ Diff. eqn. approx., $\Delta T=0.1*T_o$

Impulse Response w/ Bilinear Mapping, $\Delta T=0.1*T_o$
Fourier transform $H(jw)$ and DTFTs of diff. eqn and bilinear approximations

- Frequency (Hz)
- Magnitude (dB)

- Frequency (Hz)
- Phase (Degrees)
Both the differential equation and bilinear mapping approximations with a sampling period of \(0.001T_0\) and \(0.01T_0\) mirror the continuous time transfer function very closely, with those with a period of \(0.001T_0\) coming the closest (more apparent if you zoom in at a closer scale).
Note that the smaller the sample period, the wider the range of frequencies the discrete time filter approximates the continuous time version.
See magnified version of this region on the next page.

- Continuous time: 
- Diff. Eqn., .001*T₀: 
- Bilinear, .001*T₀: 
- Diff. Eqn., .01*T₀: 
- Bilinear, .01*T₀: 
- Diff. Eqn., .1*T₀: 
- Bilinear, .1*T₀: 

(0.9446 Hz, 0.7072) and (0.9895 Hz, 0.6442) are marked points on the graph.
Note that the differential equation approximation with sample period 0.001*T₀ is the best approximation to the continuous time group delay.
Summary: The zero at (0,0) in the continuous time version maps to the zero at $z=\exp(j0)=1$ in the discrete time pole/zero plots. Since $H(s) \rightarrow 0$ as $s \rightarrow \infty$, there is another zero at infinity. This zero maps to (-1,0) in the $z$ plane. The two poles in $H(s)$ at -1.57±j6.08 correspond to the two poles in each of the discrete transfer functions. The poles move due to the change in the sampling period, and their positions agree with the normalized frequencies shown in the transfer function plots.
% MATLAB code for Problem 4

clear all;
close all;

fo=1;           % Resonant Frequency
wo=2*pi*fo;
To=2*pi/wo;     % Resonant Period
d=.25;          % Damping Coefficient
dt=[.001 .01 .1];
DT=To*dt;
Tf=5;
NumFigures=8;

s = tf('s');
Hs = (s*wo)/(s^2 + 2*d*wo*s +wo^2);
figure(1);
subplot(4,2,1);
impulse(Hs,Tf)
title(['Impulse response of H(s) w/ f_o=',num2str(fo),', \delta=',num2str(d)])

figure(2);
pHs=pole(Hs);
zHs=zero(Hs);
subplot(4,1,1);
plot(real(pHs),imag(pHs),'xr',real(zHs),imag(zHs),'ob');
xlabel('Real axis');
ylabel('Imaginary axis');
title('Pole/Zero plot for H(s)');
axis([-2 2 -7 7]);

figure(4);
f=linspace(0,10,1e4);
s1=j*2*pi*f;
% H = freqs(wo,[1 2*d*wo wo^2],2*pi*f);
H = (s1*wo)./(s1.^2 + 2*d*wo*s1 +wo^2);
subplot(211);
plot(f,20*log10(abs(H)),'k');
hold on;
subplot(212);
plot(f,phase(H)*180/pi,'k');
hold on;

figure(3);
f2=linspace(0,3,10000);
omega=2*pi*f2;
group=((1/(d*wo))+(wo^2-omega.^2)./(2*d*wo*omega.^2))./(1+((wo^2-omega.^2).^2)./(4*(d^2)*
(wo^2)*(omega.^2)));
plot(f2,group,'k');
color=['g';'r';'m';'y';'b';'c'];
for k=1:3,
z = tf('z',DT(k));
a1=1/DT(k)+d*wo;
a2=DT(k)*wo^2-2/DT(k);
a3=1/DT(k)-d*wo;
b=wo/2;

figure(1);
Hz=(b*(z-z^(-1)))/(a1*z+a2+a3*(z^(-1)));
subplot(4,2,2*(k+1)-1);
impulse(Hz,Tf)
title(['Impulse Response w/ Diff. eqn. approx., \Delta T=',num2str(dt(k)),'*T_o']);
axis([0 Tf -6.3*dt(k) 6.3*dt(k)]);
s2=(2/DT(k))*((z-1)/(z+1));
Hbilin = (s2*wo)/(s2^2 + 2*d*wo*s2 +wo^2);
subplot(4,2,2*(k+1));
impulse(Hbilin,Tf)
title(['Impulse Response w/ Bilinear Mapping, \Delta T=',num2str(dt(k)),'*T_o']);
axis([0 Tf -6.3*dt(k) 6.3*dt(k)]);

figure(2);
pHz=pole(Hz);
zHz=zero(Hz);
pHbilin=pole(Hbilin);
zHbilin=zero(Hbilin);
subplot(4,2,2*(k+1)-1);
zplane(zHz,pHz)
axis([-1.1 1.1 -1.1 1.1]);
axis square;
title(['Diff. eqn. approx., \Delta T=',num2str(dt(k)),'*T_o']);
subplot(4,2,2*(k+1));
zplane(zHbilin,pHbilin)
axis([-1.1 1.1 -1.1 1.1]);
axis square;
title(['Bilinear Mapping, \Delta T=',num2str(dt(k)),'*T_o']);

[bHz,aHz] = tfdata(Hz,'v');
[bHbilin,aHbilin] = tfdata(Hbilin,'v');

figure(4);
subplot(212);
[H_Hz,f_Hz]=freqz(bHz,aHz,le4,1/DT(k));
hold on;
[H_bilin,f_bilin]=freqz(bHbilin,aHbilin,le4,1/DT(k));
subplot(211);
plot(f_Hz,20*log10(abs(H_Hz)),color(2*(k)-1));
hold on;
plot(f_bilin,20*log10(abs(H_bilin)),color(2*(k)));
hold on;
xlabel('Frequency (Hz)');
ylabel('Magnitude (dB)');
subplot(212);
plot(f_Hz,phase(H_Hz)*180/pi,color(2*(k)-1));
hold on;
plot(f_bilin,phase(H_bilin)*180/pi,color(2*(k))); hold on;
xlabel('Frequency (Hz)');
ylabel('Phase (Degrees)');

figure(3);
hold on;
[bHz,aHz] = tfdata(Hz,'v');
[Gd_Hz,F_Hz]=grpdelay(bHz,aHz,1e4,1/DT(k));
plot(F_Hz,Gd_Hz*DT(k),color(2*(k)-1));
hold on;
[bHbilin,aHbilin] = tfdata(Hbilin,'v');
[Gd_Hbilin,F_Hbilin]=grpdelay(bHbilin,aHbilin,1e4,1/DT(k));
plot(F_Hbilin,Gd_Hbilin*DT(k),color(2*(k))); 
end

gfigure(3);
legend('Continuous time','Diff. Eqn., .001*T_o','Bilinear, .001*T_o','Diff. Eqn., .01*T_o','Bilinear, .01*T_o');
axis([0 3 0 .75])
ylabel('Group Delay (seconds)');
xlabel('Frequency (Hz)');

figure(4);
legend('H(s)','Diff. Eqn., .001*T_o','Bilinear, .001*T_o','Diff. Eqn., .01*T_o','Bilinear, .01*T_o');
axis([0 5 -70 7]);
subplot(211);
title('Fourier transform H(jw) and DTFTs of diff. eqn and bilinear approximations');
axis([0 5 -100 100]);
subplot(212);
axis([0 5 -100 100]);
Problem 4. i) \( W(t) = e^{-\frac{t}{2} \left( \frac{c}{M} \right)^2} + e^{-\frac{t}{2} \left( \frac{c}{M} \right)^2} \left( \frac{c}{T_H} \right)^2 \)

\[ \mathcal{F}\left[ e^{-\frac{t}{2} \left( \frac{c}{M} \right)^2} \right] = \sqrt{2\pi} M e^{-\frac{e^{-\frac{t}{2} \left( \frac{c}{M} \right)^2}}{2 \pi M^2 f^2}} \]

\[ \frac{d}{dt} e^{-\frac{t}{2} \left( \frac{c}{M} \right)^2} = -\left( \frac{c}{M^2} \right) e^{-\frac{t}{2} \left( \frac{c}{M} \right)^2} \]

We know that \( \mathcal{F}\left[ \frac{d}{dt} f(t) \right] = (j2\pi f) \mathcal{F}\{f(t)\} \), so

\[ \mathcal{F}\left[ -\left( \frac{c}{M^2} \right) e^{-\frac{t}{2} \left( \frac{c}{M} \right)^2} \right] = j(2\pi)^{\frac{3}{2}} M f e^{-\frac{e^{-\frac{t}{2} \left( \frac{c}{M} \right)^2}}{2 \pi M^2 f^2}} \]

We also know that

\[ \mathcal{F}\left[ t f(t) \right] = \frac{j}{2\pi} \frac{d}{df} \left[ \mathcal{F}\{f(t)\} \right] \], so

\[ \mathcal{F}\left[ -\frac{t^2}{M^2} e^{-\frac{t}{2} \left( \frac{c}{M} \right)^2} \right] = -\sqrt{2\pi} M \left( e^{-2\pi^2 M f^2} - 4\pi M f \right) e^{-\frac{e^{-\frac{t}{2} \left( \frac{c}{M} \right)^2}}{2 \pi M^2 f^2}} \]

\[ = -\sqrt{2\pi} M \left(1 - 4\pi M f^2 \right) e^{-\frac{e^{-\frac{t}{2} \left( \frac{c}{M} \right)^2}}{2 \pi M^2 f^2}} \]

\[ \downarrow \]

\[ \mathcal{F}\left[ \left( \frac{c}{T_H} \right)^2 e^{-\frac{t}{2} \left( \frac{c}{M} \right)^2} \right] = \sqrt{2\pi} \frac{M^3}{T_H} e^{-\frac{e^{-\frac{t}{2} \left( \frac{c}{M} \right)^2}}{2 \pi M^2 f^2}} \left(1 - 4\pi M f^2 \right) \]

So

\[ W(f) = \sqrt{2\pi} M e^{-2\pi M f^2} \left( 1 + \frac{M^2}{T_H^2} - \frac{M^2}{T_H^2} \frac{4\pi^2 M^2 f^2}{M^2} \right) \]
We know that \( \mathbb{E}[\hat{s}_x(f|t)] = \int_{-\infty}^{\infty} S_x(v) W(f-v) dv = S_x(f) * W(f) \).

ii) Let's observe \( W(f) \):
\[
\exp\left(-\frac{\pi M^2 f^2}{T_h^2}\right) \approx \frac{3}{5} f^{-3} \quad \text{at} \quad f = \frac{1}{M}
\]
\[
\left(1 + \frac{M^2}{T_h^2} - \frac{M^2}{T_h^2} \frac{4 \pi^2 M^2 f^2}{T_h^2}\right) \text{ reaches 0 at } f_0 = \frac{1}{2\pi M} \sqrt{1 + \frac{T_h^2}{M^2}} = \frac{\sqrt{M^2 + T_h^2}}{2\pi M^2}
\]

\[
f_0 \approx \left\{ \begin{array}{ll}
\frac{T_h}{2\pi M^2} & \gg \frac{1}{M} \quad T_h \gg M \\
\frac{1}{2\pi M} & \text{ } \quad T_h < M
\end{array} \right.
\]

So we see that generally, the bandwidth of \( W(f) \) is \( \approx \frac{1}{M} \), in spite of the value of \( T_h \).

Now let's see the resolution in estimating the boxcar spectrum:

1. \( W \gg \frac{1}{M} \), i.e., \( WM \gg 1 \), resolution is \( \frac{1}{M} \)

2. \( W \approx \frac{1}{M} \), i.e., \( WM \approx 1 \),

\[
\begin{align*}
-\frac{1}{M} & \quad \frac{1}{M} \quad * \quad \text{(convolution)} \\
-\frac{1}{M} & \quad \frac{1}{M}
\end{align*}
\]

3. \( W \ll \frac{1}{M} \), i.e., \( WM \ll 1 \)

\[
\begin{align*}
-\frac{1}{M} & \quad \frac{1}{M} \quad * \quad \text{(re-scaled but same bandwidth as } W(f) )
\end{align*}
\]

There's no much point in talking about estimation resolution because the output spectrum estimate is completely smeared out by \( W(f) \).
iii) \( \sigma_3^2 (\xi | \tau) = \frac{S_\xi (\xi)}{T} \int_{-\tau}^{\tau} W(v) \, dv \)

By Parseval's theorem,

\[
\int_{-\tau}^{\tau} W(v) \, dv = \int_{-\tau}^{\tau} W^2(\tau) \, d\tau
\]

\[
= \int_{-\tau}^{\tau} e^{-(\frac{\tau}{M})^2} \left[ 1 + 2\frac{\tau^2}{T_H^2} + \frac{\tau^4}{T_H^4} \right] \, d\tau
\]

Note that \( \int_{-\tau}^{\tau} e^{-\frac{u^2}{2}} \, du = \sqrt{2\pi} \int_{-\tau}^{\tau} u e^{-\frac{u^2}{2}} \, du = \sqrt{2\pi} \), and then

\[
\int_{-\tau}^{\tau} u^4 e^{-\frac{u^2}{2}} \, du = 3 \sqrt{2\pi} \quad \text{and then}
\]

\[
\int_{-\tau}^{\tau} e^{-\frac{\tau^2}{2}} \, d\tau = \sqrt{\pi} M
\]

\[
\int_{-\tau}^{\tau} 2 \frac{\tau^2}{T_H^2} e^{-\frac{\tau^2}{2}} \, d\tau = \frac{M^3}{T_H^2} \sqrt{\pi}
\]

\[
\int_{-\tau}^{\tau} 4 \frac{\tau^4}{T_H^4} e^{-\frac{\tau^2}{2}} \, d\tau = \frac{3}{4} \frac{M^5}{T_H^4} \sqrt{\pi}
\]

\[
\implies \int_{-\tau}^{\tau} W^2(\tau) \, d\tau = \int_{-\tau}^{\tau} W^2(v) \, dv = \sqrt{\pi} M \left[ 1 + \left( \frac{M}{T_H} \right)^2 + \frac{3}{4} \left( \frac{M}{T_H} \right)^4 \right]
\]

\[
\frac{\sigma_3^2 (\xi | \tau)}{S_\xi^2 (\xi)} = C_w \frac{M}{T}
\]

where \( C_w = \sqrt{\pi} \left[ 1 + \left( \frac{M}{T_H} \right)^2 + \frac{3}{4} \left( \frac{M}{T_H} \right)^4 \right] \)