PS3 Solution

Problem 1

1) the detection metric is given as:

$$d^2 = \frac{\left(E(L|H_1) - E[L|H_0]\right)^2}{\sigma^2_{L|H_0}}$$

$$E(L|H_1) = E\left[\|\underline{s}+\underline{w}\|^2\right] = E\left[(\underline{s}^H+\underline{w}^H)(\underline{s}+\underline{w})\right] =$$

$$= E\left[\underline{s}^H\underline{s} + \underline{s}^H\underline{w} + \underline{w}^H\underline{s} + \underline{w}^H\underline{w}\right] = E\left[\|\underline{s}\|^2\right] + E[\underline{s}^H]\,E[\underline{w}] + E[\underline{w}^H]\,E[\underline{s}] + E\left[\|\underline{w}\|^2\right]$$

since $\underline{w}$ and $\underline{s}$ are independent.

Then $E[L|H_1] = E\left[\|\underline{s}\|^2\right] + E\left[\|\underline{w}\|^2\right]$

$$= E\left[|s_1|^2 + \cdots + |s_N|^2\right] + E\left[|w_1|^2 + \cdots + |w_N|^2\right] =$$

$$= \sum_{i=1}^{N} E[|s_i|^2] + \sum_{i=1}^{N} E[|w_i|^2] = \sum_{i=1}^{N} \lambda_i + N\sigma_w^2$$

$$E[L|H_0] = E\left[\|\underline{w}\|^2\right] = N\sigma_w^2$$

$$\sigma^2_{L|H_0} = E[L^2|H_0] - E[L|H_0]^2 = E[L^2|H_0] - (N\sigma_w^2)^2$$

$$E[L^2|H_0] = E\left[\|\underline{r}\|^4\right] = E\left[\|\underline{w}\|^4\right] = E\left[(\underline{w}^H\underline{w})^2\right] =$$

$$= E\left[\sum_{i=1}^{N} w_i^* w_i \sum_{j=1}^{N} w_j^* w_j\right] = E\left[\sum_i \sum_j |w_i|^2 |w_j|^2\right]$$

If $i=j$ then $E[L^2|H_0] = \sum_{i=1}^{N} E\left[|w_i|^4\right] = \sum_{i=1}^{N} E\left[w_i^* w_i w_i^* w_i\right]$

from the moment thm for Gaussians we have

$E[w_i^* w_i w_i^* w_i] = 2\sigma_w^4$ hence $E[L^2|H_0] = 2N\sigma_w^4$

If $i \neq j$ then $E[L^2|H_0] = \sum_i \sum_{j,\, i\neq j} E[|w_i|^2]\,E[|w_j|^2] = (N^2-N)\sigma_w^4$

Hence, $E[L^2/H_0] = (N^2 + N)\sigma_w^4$

and $\sigma^2_{L/H_0} = E[L^2/H_0] - E^2[L/H_0] = N\sigma_w^4$

So the detection metric is

$$d^2 = \frac{\left(\sum_{i=1}^{N} \lambda_i + N\sigma_w^2 - N\sigma_w^2\right)^2}{N\sigma_w^4} = \frac{\left(\sum_{i=1}^{N} \lambda_i\right)^2}{N\sigma_w^4}$$

ii) if we have a power constraint (this happens always in practice) then

$$d^2 = \frac{E_s^2}{N\sigma_w^4}$$ which is a signal-to-noise ratio.

Obviously to maximize d^2 we better choose $N = 1$. This means it is better to put all the power to one signal (s_i) than to distribute the available power among different signals $s_1, \ldots, s_N$ because the resulting SNR per signal $\left(\frac{\lambda_i^2}{\sigma_w^4}\right)$ is higher when $\lambda_i = E_s$.

Problem 2

i) $E[y(t)] = E[s(t)a(t) + (1-s(t))b(t)]$ $\overset{\text{uncorrelated processes}}{=}$

$= E[s(t)]E[a(t)] + (1 - E[s(t)])E[b(t)]$

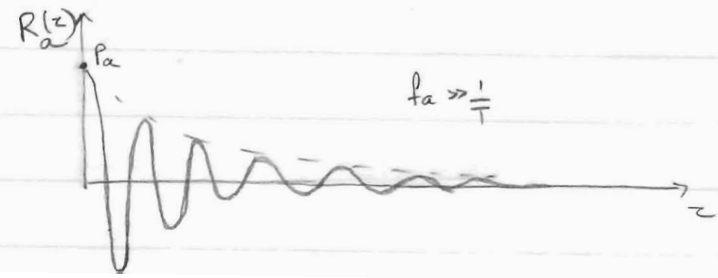
We know that $R_a(\tau) = P_a \cos(2\pi f_a \tau) e^{-|\tau|/T}$

$= E[a(t+\tau)\, a^*(t)]$

Note that $\lim_{\tau \to \infty} E[a(t+\tau)\, a^*(t)] = E[a(t+\tau)]\, E[a^*(t)]$

$a(t)$ is WSS so $E[a(t)]$ is const

$= |m_a|^2$ in other words the process decorrelates when $\tau \to \infty$ as the graph shows:



Thus $|m_a|^2 = \lim_{\tau \to \infty} \left| P_a \cos(2\pi f_a \tau) e^{-|\tau|/T} \right| \leq \lim_{\tau \to \infty} P_a \left| e^{-|\tau|/T} \right|$

$= 0$ hence $m_a = 0$

Similarly, $m_b = 0$.

Hence, $E[y(t)] = 0$.

$E[y(t+\tau)\, y^*(t)] = E\left[\left(s(t+\tau)\, a(t+\tau) + (1-s(t+\tau))\, b(t+\tau)\right)\left(s^*(t)\, a^*(t) + (1-s^*(t))\, b^*(t)\right)\right]$

since all signals are uncorellated we get

$E[y(t+\tau)\, y^*(t)] = R_s(\tau) R_a(\tau) + \left[E[s(t+\tau)] - R_s(\tau)\right] E[a(t+\tau)]\, E[b^*(t)] +$

$\left(-R_s(\tau) + E[s^*(t)]\right) E[a^*(t)]\, E[b(t+\tau)] + R_b(\tau)\left(1 - E[s(t)] - E[s^*(t+\tau)] + R_s(\tau)\right)$

$= R_s(\tau) R_a(\tau) + \left(m_s - R_s(\tau)\right) m_a m_b^* + \left(-R_s(\tau) + m_s^*\right) m_a^* m_b + R_b(\tau)\left(1 - m_s - m_s^* + R_s(\tau)\right)$

$|m_s|^2 = \lim_{\tau \to \infty} \frac{1}{4}\left(e^{-\lambda|\tau|} + 1\right) = \frac{1}{4} \Rightarrow m_s = \frac{1}{2}$ ($s(t)$ is real)

Thus $E[y(t+\tau)y^*(t)] = R_s(\tau)R_a(\tau) + R_s(\tau)R_b(\tau) =$

$= R_s(\tau)\left(R_a(\tau) + R_b(\tau)\right)$

$= \frac{1}{4}\left(e^{-\lambda|\tau|}+1\right)\left(\cos(2\pi f_a\tau)+\cos(2\pi f_b\tau)\right)P_a e^{-|\tau|/T} = R_y(\tau)$

Obviously $y(t)$ is WSS.

Recall that $e^{-a|\tau|} \overset{F}{\longleftrightarrow} \frac{2a}{(2\pi f)^2 + a^2}$

$$R_y(\tau) = \frac{P_a}{4}\left(e^{-|\tau|\left(\lambda+\frac{1}{T}\right)} + e^{-|\tau|/T}\right)\left(\cos(2\pi f_a\tau) + \cos(2\pi f_b\tau)\right)$$

$$\Rightarrow S_y(f) = \frac{P_a}{4}\left(\frac{2\left(\lambda+\frac{1}{T}\right)}{(2\pi f)^2 + \left(\lambda+\frac{1}{T}\right)^2} + \frac{2\,1/T}{(2\pi f)^2 + 1/T^2}\right) \circledast \frac{1}{2}\left(\begin{array}{l}\delta(f-f_a)+\delta(f-f_b)\\ +\delta(f+f_a)+\delta(f+f_b)\end{array}\right)$$

$$= \frac{P_a}{4}\left[\frac{\lambda+1/T}{(2\pi(f-f_a))^2+\left(\lambda+\frac{1}{T}\right)^2} + \frac{1/T}{(2\pi(f-f_a))^2+1/T^2} + \frac{\lambda+1/T}{(2\pi(f-f_b))^2+\left(\lambda+\frac{1}{T}\right)^2} + \frac{1/T}{(2\pi(f-f_b))^2+\frac{1}{T^2}}\right.$$

$$\left. + \frac{\lambda+1/T}{(2\pi(f+f_a))^2+\left(\lambda+\frac{1}{T}\right)^2} + \frac{1/T}{(2\pi(f+f_a))^2+1/T^2} + \frac{\lambda+1/T}{(2\pi(f+f_b))^2+\left(\lambda+\frac{1}{T}\right)^2} + \frac{1/T}{(2\pi(f+f_b))^2+1/T^2}\right]$$

ii) The figure below shows the plot of $S_y(f)$ in dB

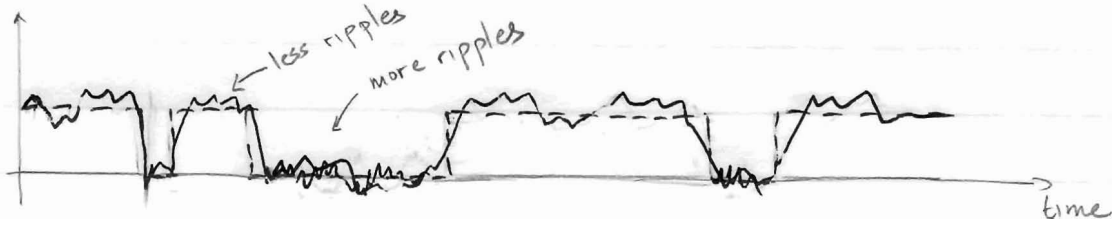
I used: $f_b = 1000$ Hz

$f_a = 100$ Hz

$T = 1$ sec

$\lambda = 10^{-2}$

A sample function of the ensemble would look like



Problem 3

$$\sum_{k=0}^{N} a_k\, y[n-k] = \sum_{k=0}^{M} b_k\, x[n-k]$$

Here, $M = N = 4$

$b = [0.0675 \quad 0.0000 \quad -0.1349 \quad 0.0000 \quad 0.0675]$

$a = [1.0000 \quad -1.9425 \quad 2.1192 \quad -1.2167 \quad 0.4128]$

(i) Plot $H(e^{j\omega})$

★ See attached plots

(ii) Plot the phase & group delays vs. frequency

★ See attached plots

(iii) Implement the difference eqn when $x[n] = \delta[n]$

$\text{Max}[|y[n]|] = 0.2409$

1% of max response $= 0.0024$

For $n \geq 27$ $|y[n]| < 0.0024$

★ See attached plot of the difference equation response to a unit impulse sequence (i.e. impulse response)

(iv) Determine the poles & zeros of the difference equation

Note: $$H(e^{j\omega}) = \frac{b_1 + b_2 e^{-j\omega} + b_3 e^{-j2\omega} + b_4 e^{-j3\omega} + b_5 e^{-j4\omega}}{a_1 + a_2 e^{-j\omega} + a_3 e^{-j2\omega} + a_4 e^{-j3\omega} + a_5 e^{-j4\omega}}$$

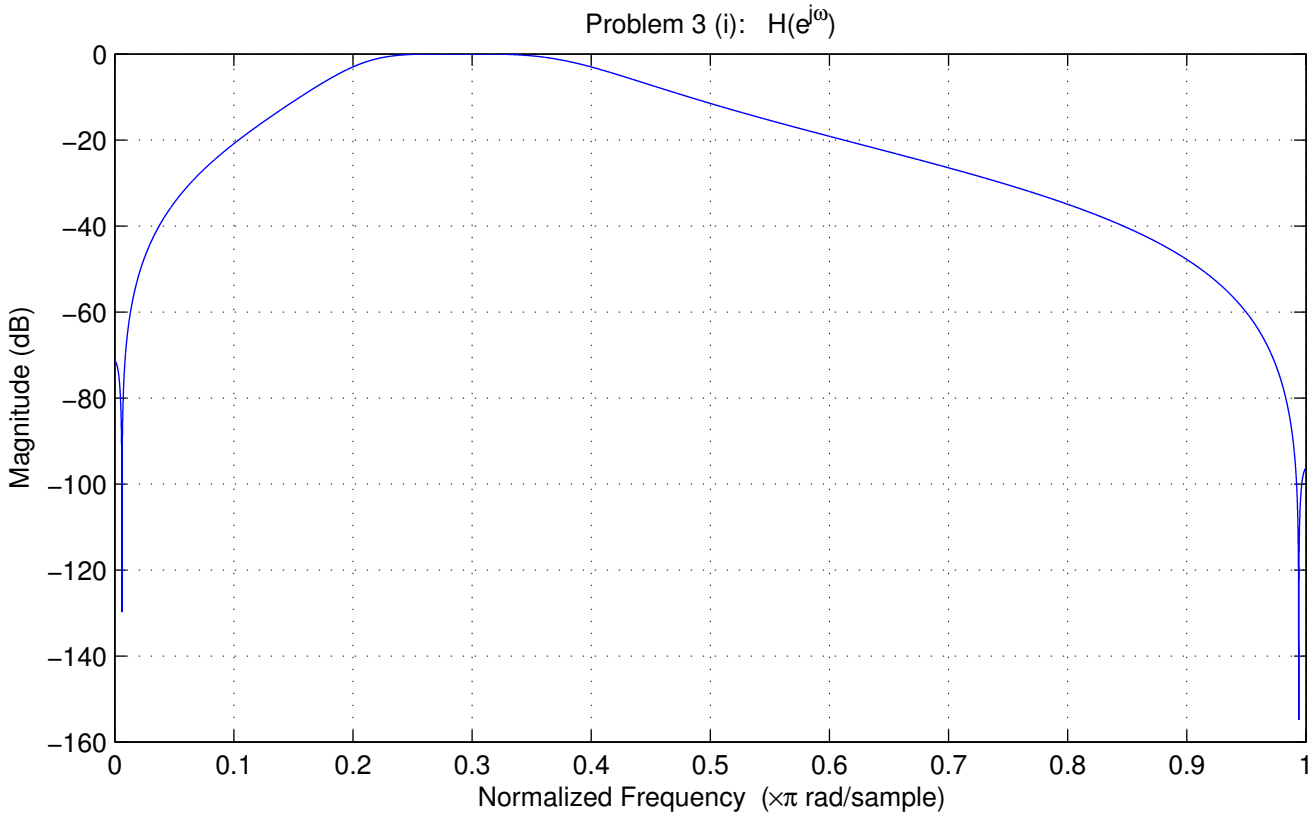
★ See the attached pole/zero plot

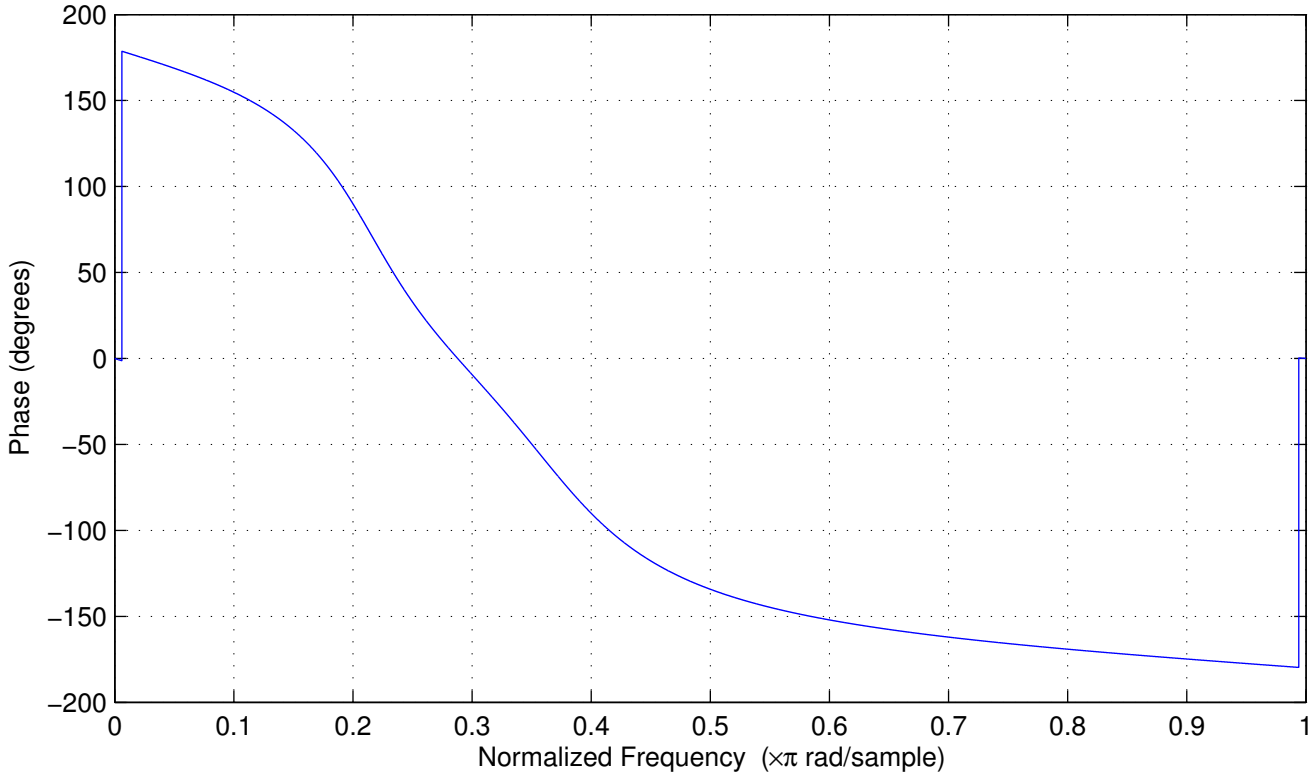
Poles: $0.6540 \pm i\,0.5193$

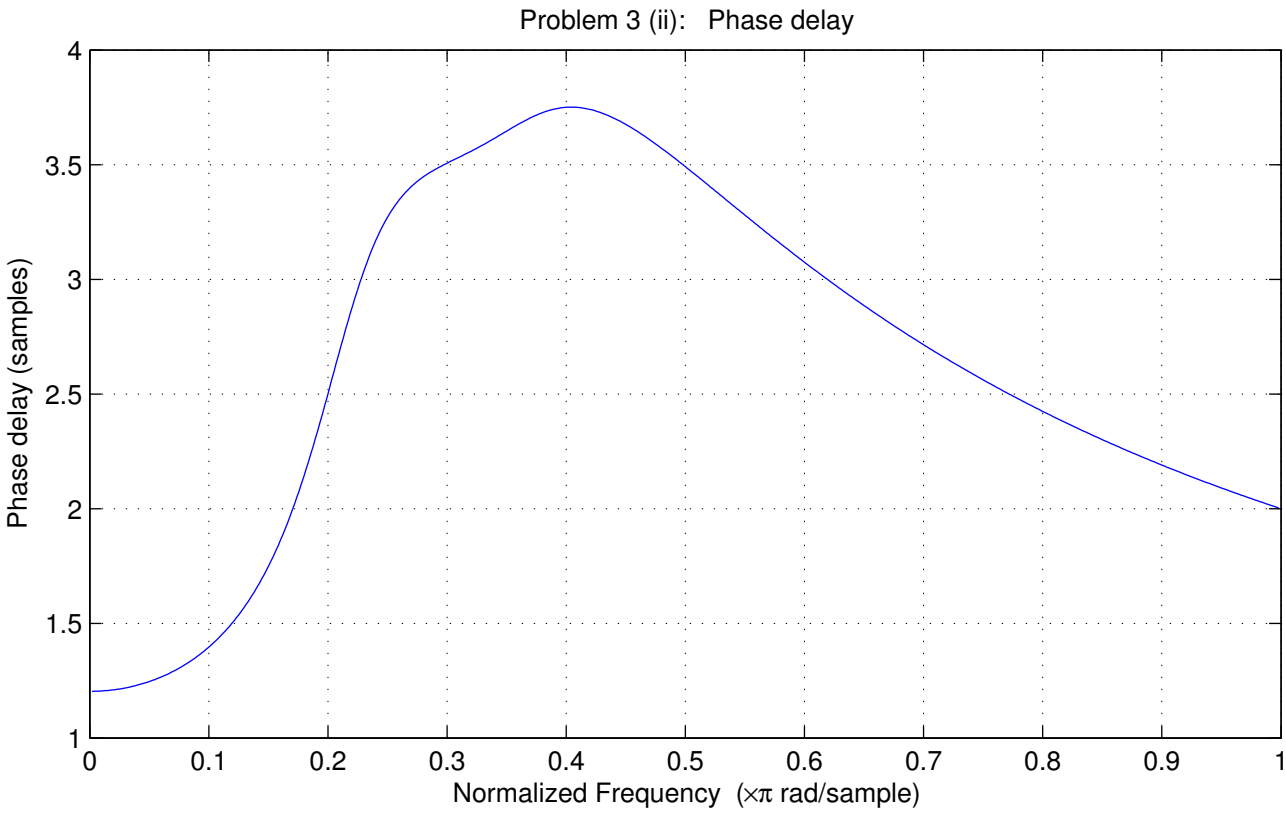
$0.3172 \pm i\,0.7009$

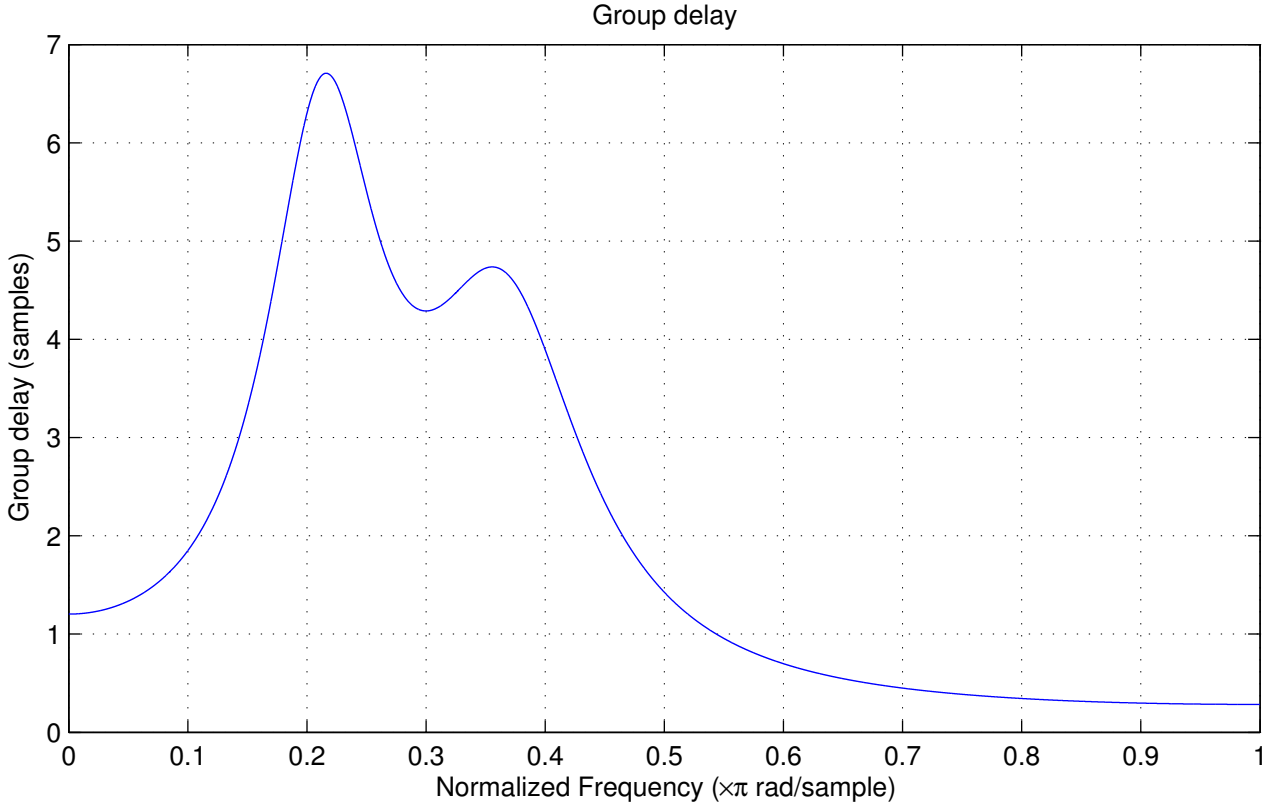
Zeros: $0.9998 \pm i\,0.0192$

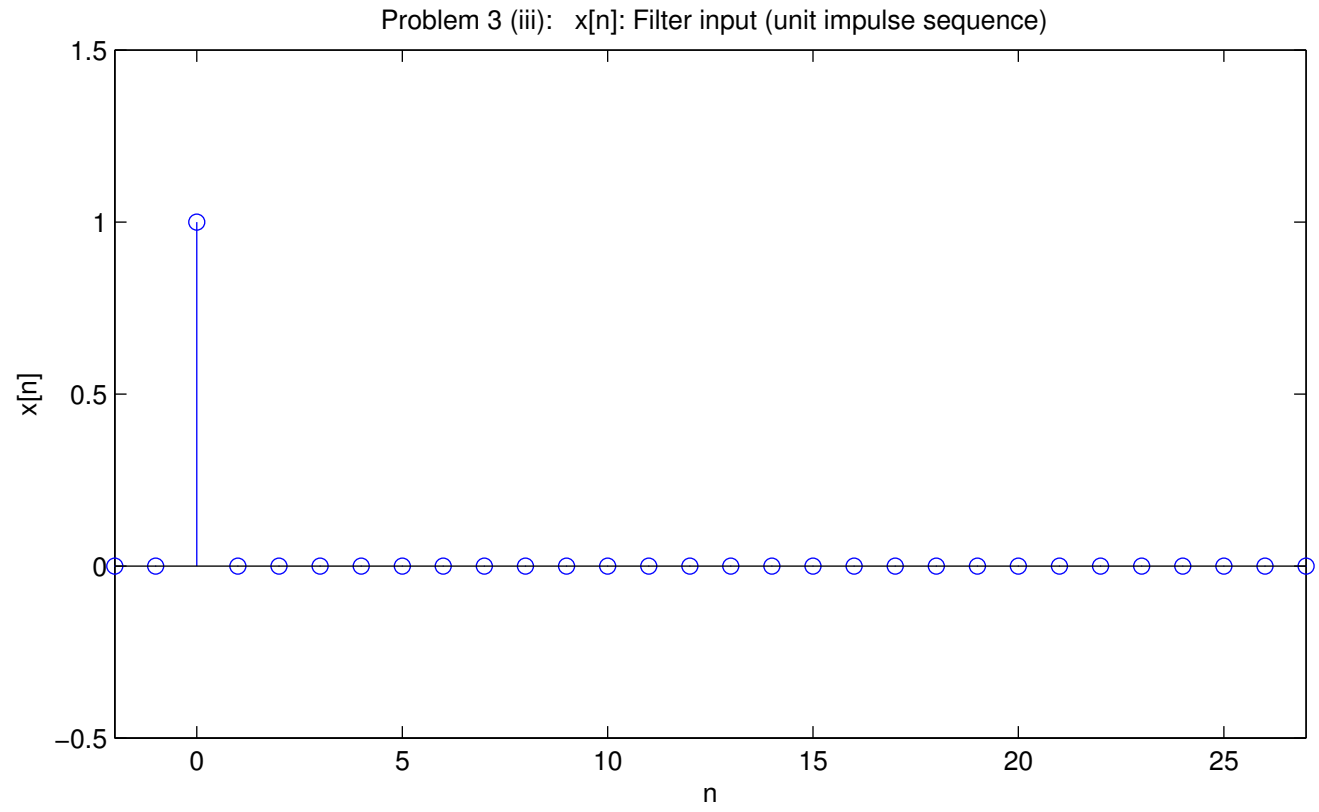
$-0.9998 \pm i\,0.0192$

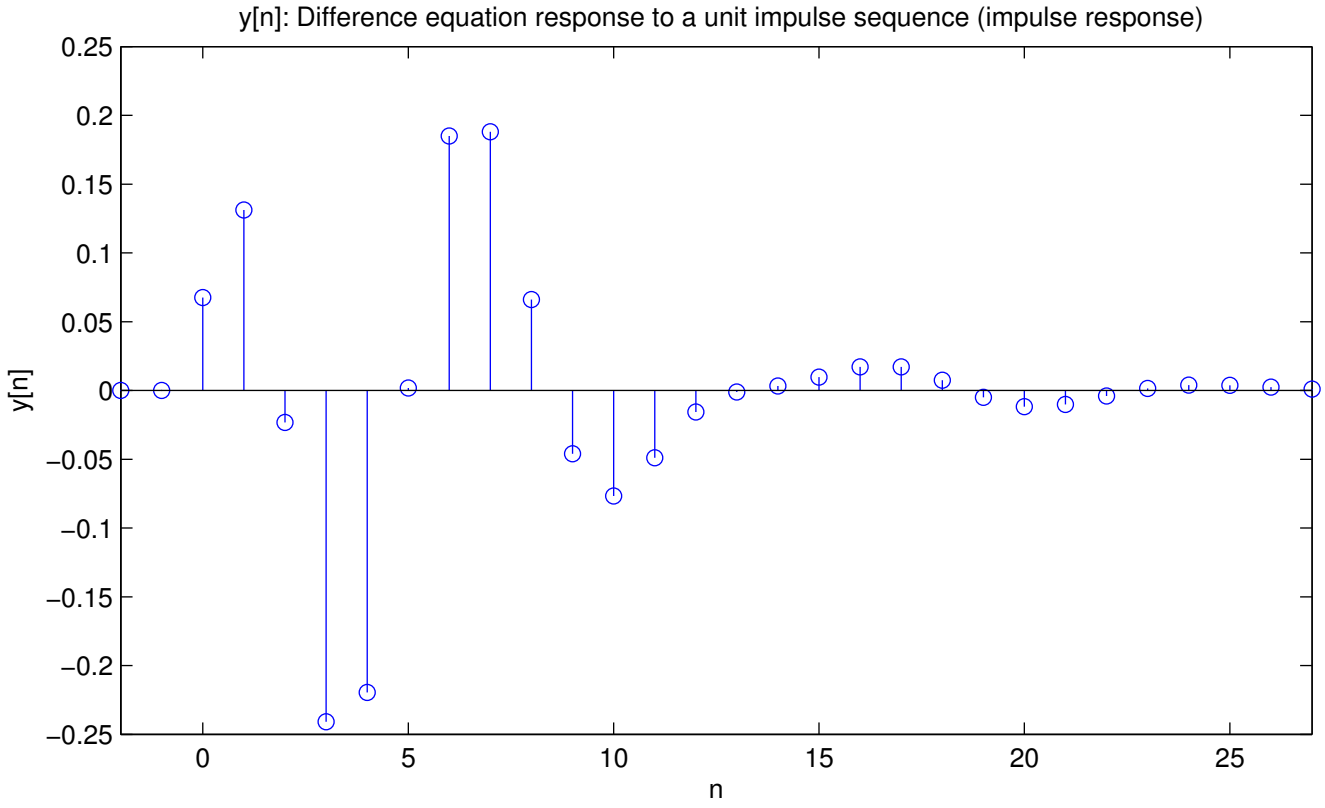

Problem 3 (i): H(e^jω)
Magnitude (dB)
0
−20
−40
−60
−80
−100
−120
−140
−160
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Normalized Frequency (×π rad/sample)

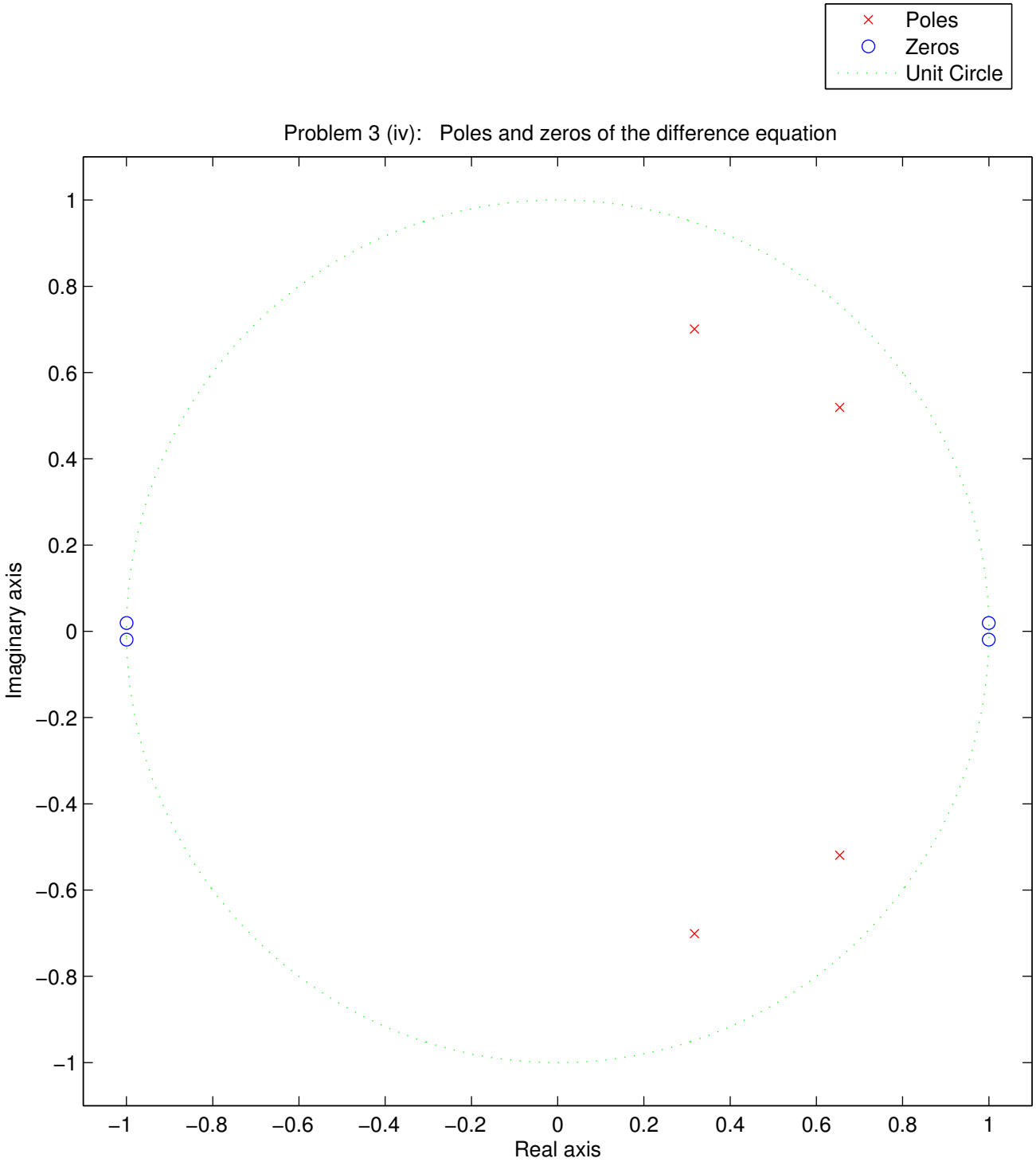

Phase (degrees)
200
150
100
50
0
−50
−100
−150
−200
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Normalized Frequency (×π rad/sample)


Problem 3 (ii): Phase delay
Phase delay (samples)
Normalized Frequency (×π rad/sample)
4
3.5
3
2.5
2
1.5
1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


Group delay
Group delay (samples)
Normalized Frequency (×π rad/sample)
7
6
5
4
3
2
1
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


Problem 3 (iii): x[n]: Filter input (unit impulse sequence)
x[n]
n
1.5
1
0.5
0
−0.5
0
5
10
15
20
25


y[n]: Difference equation response to a unit impulse sequence (impulse response)
y[n]
n
0.25
0.2
0.15
0.1
0.05
0
−0.05
−0.1
−0.15
−0.2
−0.25
0
5
10
15
20
25


Poles
Zeros
Unit Circle
Problem 3 (iv): Poles and zeros of the difference equation
Imaginary axis
Real axis
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1

```
%Code for Problem Set 3, Problem 3

close all;
clear all;

b=[0.0675  0.0000 -0.1349  0.0000  0.0675];
a=[1.0000 -1.9425  2.1192 -1.2167  0.4128];

% i) Plot Frequency Response)
N=100000;
figure(1)
freqz(b,a,N)
title('Problem 3 (i):   H(e^{j\omega})');

% ii) Plot the phase and group delays
figure(2)
subplot(211);
phasedelay(b,a);
title('Problem 3 (ii):   Phase delay');
subplot(212);
grpdelay(b,a,N);
title('Group delay');

% iii) Plot the impulse response
n=-2:27
x=zeros(30,1);
x(3)=1;
y=filter(b,a,x);
figure(3);
subplot(211);
stem(n,x);
xlabel('n');
ylabel('x[n]');
title('Problem 3 (iii):   x[n]: Filter input (unit impulse sequence)');
axis([-2 27, -.5 1.5]);
subplot(212);
stem(n,y);
xlabel('n');
ylabel('y[n]');
axis([-2 27, -.25 .25]);
title('y[n]: Difference equation response to a unit impulse sequence (impulse response)');

% iv) Find the poles and zeros
poles=roots(a);
zeros=roots(b);
figure(4);
theta=linspace(0,2*pi,1000);
plot(real(poles),imag(poles),'xr',real(zeros),imag(zeros),'ob',cos(theta),sin(theta),':↙
g');
legend('Poles','Zeros','Unit Circle');
axis([-1.1 1.1 -1.1 1.1]);
axis square;
```

```
xlabel('Real axis');
ylabel('Imaginary axis');
title('Problem 3 (iv):   Poles and zeros of the difference equation');
```

Problem 4 – Transforms to the discrete domain

$$H(s) = \frac{s\omega_0}{s^2 + 2\delta\omega_0 s + \omega_0^2}$$ with resonant period $T_0 = \frac{2\pi}{\omega_0}$ & resonant frequency $f_0 = \frac{1}{T_0}$

i) Determine the corresponding difference equation when the derivative operations are given by

$$\frac{\partial x(t)}{dt} \rightarrow \frac{x[n+1] - x[n-1]}{2\Delta T}$$

$$\frac{\partial^2 x(t)}{dt^2} \rightarrow \frac{x[n+1] - 2x[n] + x[n-1]}{\Delta T^2}$$

Note that the problem statement has an error in the denominator

$$Y(s)\left(s^2 + 2\delta\omega_0 s + \omega_0^2\right) = X(s)\left(s\omega_0\right)$$

$$\frac{\partial^2}{\partial t^2} y(t) + 2\delta\omega_0 \frac{\partial}{\partial t} y(t) + \omega_0^2 y(t) = \omega_0 \frac{\partial}{\partial t} x(t)$$

$$\Downarrow$$

$$\frac{y[n+1] - 2y[n] + y[n-1]}{\Delta T^2} + 2\delta\omega_0 \frac{y[n+1] - y[n-1]}{2\Delta T} + \omega_0^2 y[n] = \omega_0 \frac{x[n+1] - x[n-1]}{2\Delta T}$$

$$\frac{1}{\Delta T}\left(\frac{1}{\Delta T} + \delta\omega_0\right) y[n+1] + \frac{1}{\Delta T}\left(\omega_0^2 \Delta T - \frac{2}{\Delta T}\right) y[n] + \frac{1}{\Delta T}\left(\frac{1}{\Delta T} - \delta\omega_0\right) y[n-1] = \frac{\omega_0}{2\Delta T}\left(x[n+1] - x[n-1]\right)$$

Corresponding difference equation:

$$\left(\frac{1}{\Delta T} + \delta\omega_0\right) y[n+1] + \left(\omega_0^2 \Delta T - \frac{2}{\Delta T}\right) y[n] + \left(\frac{1}{\Delta T} - \delta\omega_0\right) y[n-1] = \frac{\omega_0}{2}\left(x[n-1] - x[n-1]\right)$$

ii)

$$H(z) = \frac{\frac{\omega_0}{2}\left(z - z^{-1}\right)}{\left(\frac{1}{\Delta T} + \delta\omega_0\right) z + \left(\omega_0^2 \Delta T - \frac{2}{\Delta T}\right) + \left(\frac{1}{\Delta T} - \delta\omega_0\right) z^{-1}}$$

Problem 4 cont...

ii) cont... See the attached plots of the impulse response for $H(s)$ and the corresponding difference equations with $\Delta T = (0.001)T_0, (0.01)T_0$ & $(0.1)T_0$. Also included are the impulse responses for the bilinear mappings to the z plane for the same values of ΔT. (All plots for $f_0 = 1$, $\delta = 0.25$)

Note that all of the impulse response functions have the same general shape, but with different amplitudes (which are a function of ΔT). For $\Delta T = 0.001\,T_0$, $0.01\,T_0$, the impulse response dies of about as quickly as that for $H(s)$, but not quite as quickly for $\Delta T = 0.1\,T_0$.

iii) ★ See the attached pole-zero, DTFT, & group delay plots plots.

Note

$$\text{group delay } H(s)\Big|_{s=j\omega} = -\frac{\partial}{\partial\omega}\angle H(j\omega) = \frac{\frac{1}{\delta\omega_0} + \frac{\omega_0^2 - \omega^2}{2\delta\omega_0\omega^2}}{1 + \frac{(\omega_0^2 - \omega^2)^2}{4\delta^2\omega_0^2\omega^2}}$$

This expression is used in the following plots for group delay.

Bilinear transformation: Replace s in $H(s)$ with $\frac{2}{\Delta T}\frac{z-1}{z+1}$

★ After analyzing the attached plots of the impulse responses, the group delay, the pole/zero loci & the transfer functions DTFTs, I conclude that the difference equation approximation with $\Delta T = 0.001\,T_0$ comes closest to the fundamental properties of the continuous system (peak response, decay rate, group delays, etc.)

cont. on next page

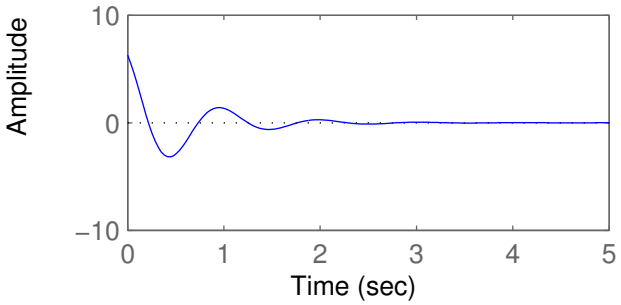
Problem 4

iii) cont...

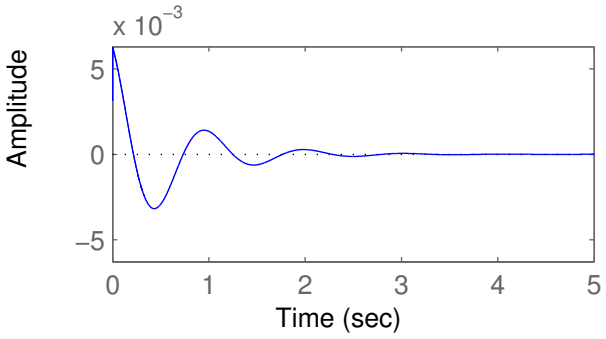
Both the bilinear & difference eqn approximations with $\Delta T = .001\, T_0$ seemed to do equally well when it came to matching $H(s)$ near the resonant frequencies. Also, the poles & zeros for these two were identical (at least to any significant resolution). Most of the impulse response functions appear to have identical decay rates, though those with $\Delta T = 0.1\, T_0$ don't decay quite as quickly as the others. The one place where the diff. eqn approx with $\Delta T = 0.001\, T_0$ had a slight better match than the bilinear mapping approx with $\Delta T = .001\, T_0$ is with the group delay (see plot). For additional details, see notes on attached plots.

22-141 50 SHEETS
22-142 100 SHEETS
22-144 200 SHEETS
AMPAD

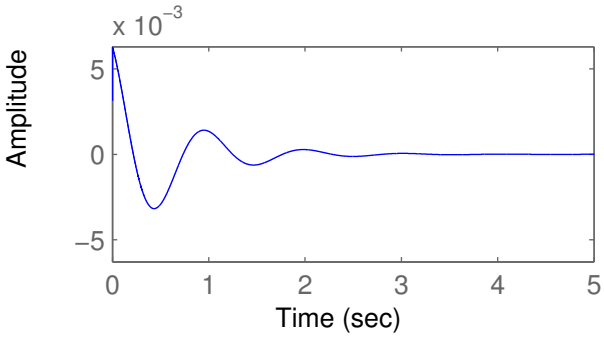
Impulse response of H(s) w/ f_o=1, δ=0.25



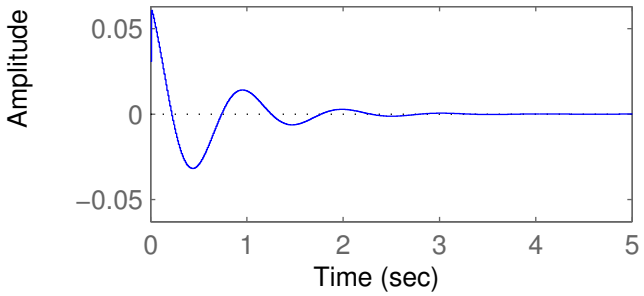
Impulse Response w/ Diff. eqn. approx., Δ T=0.001*T_o



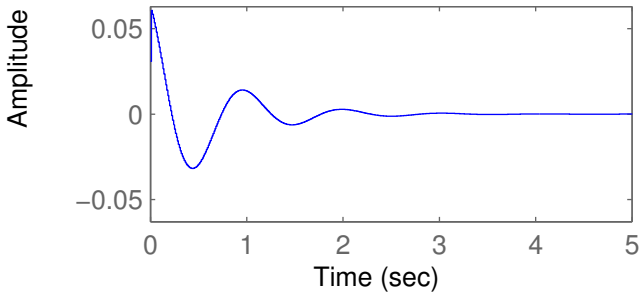
Impulse Response w/ Bilinear Mapping, Δ T=0.001*T_o



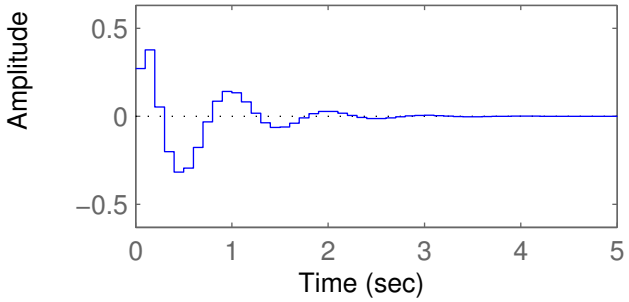
Impulse Response w/ Diff. eqn. approx., Δ T=0.01*T_o



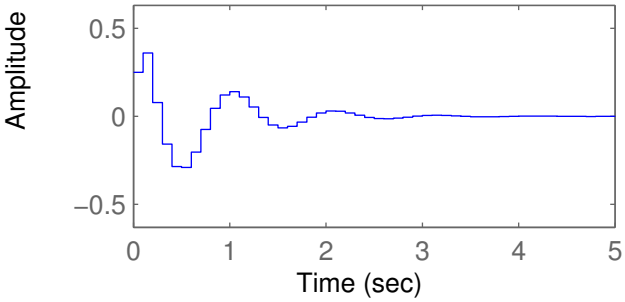
Impulse Response w/ Bilinear Mapping, Δ T=0.01*T_o

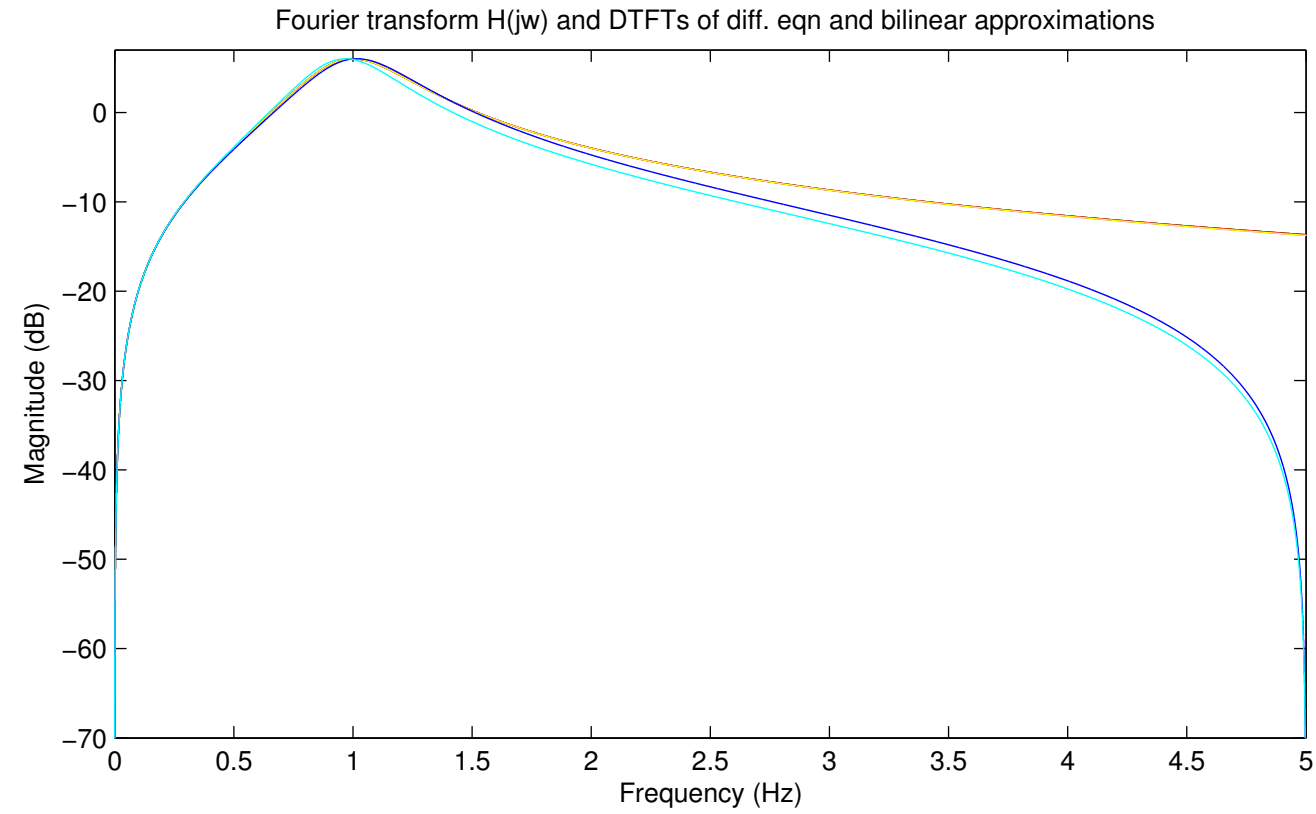


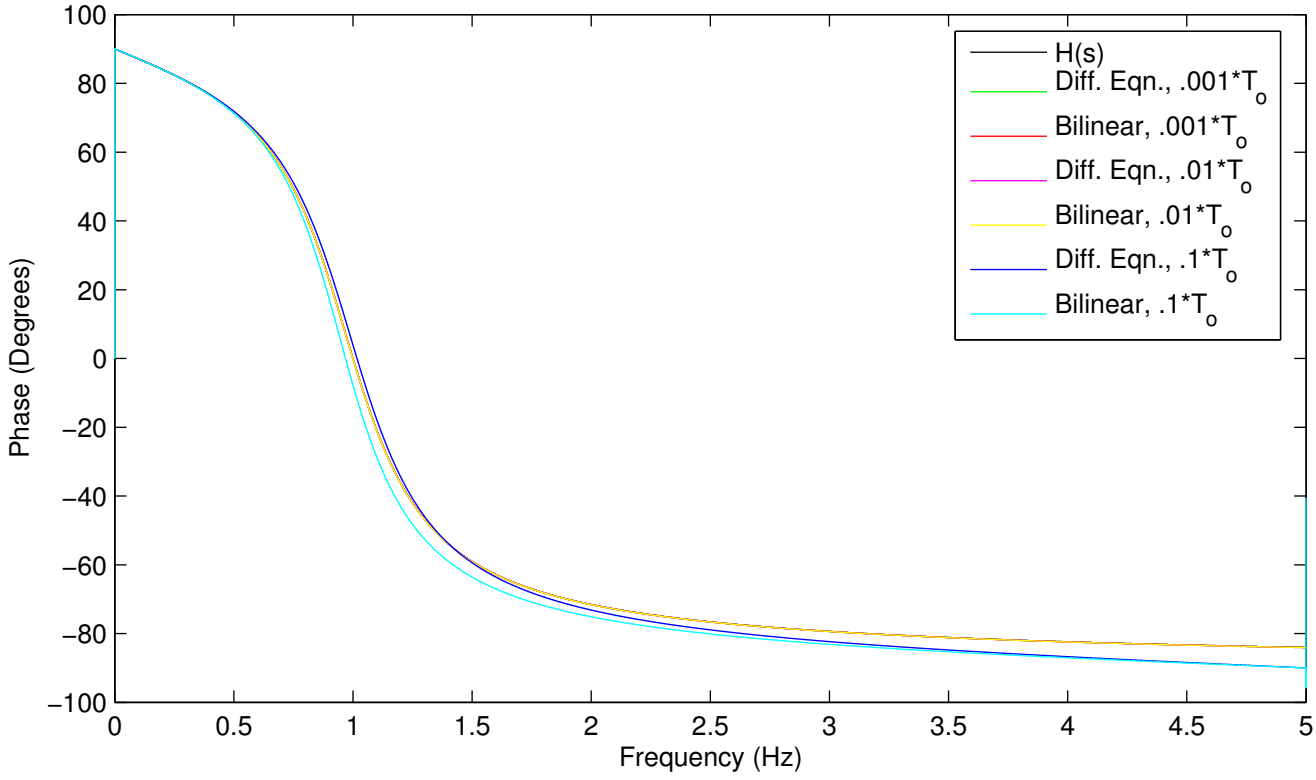
Impulse Response w/ Diff. eqn. approx., Δ T=0.1*T_o



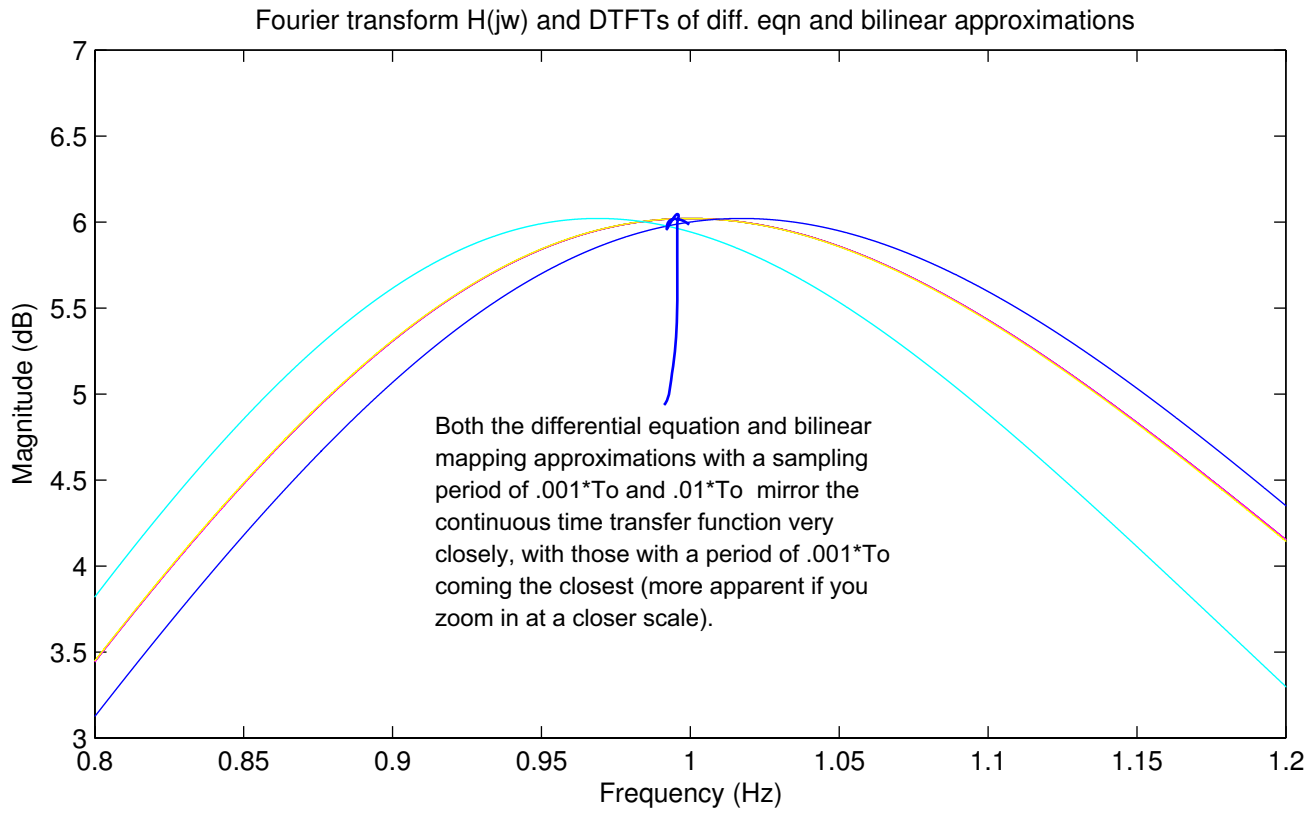
Impulse Response w/ Bilinear Mapping, Δ T=0.1*T_o

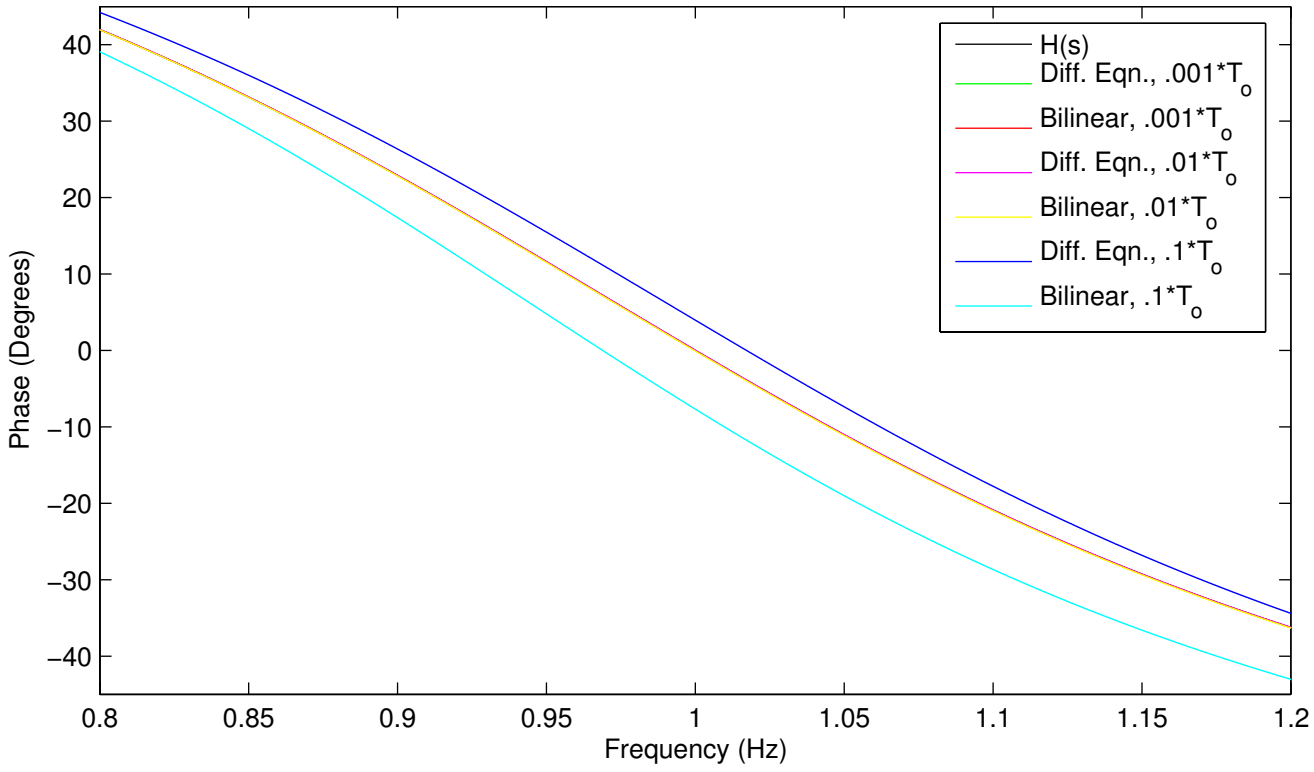



Fourier transform H(jw) and DTFTs of diff. eqn and bilinear approximations
Magnitude (dB)
Frequency (Hz)
0
−10
−20
−30
−40
−50
−60
−70
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5

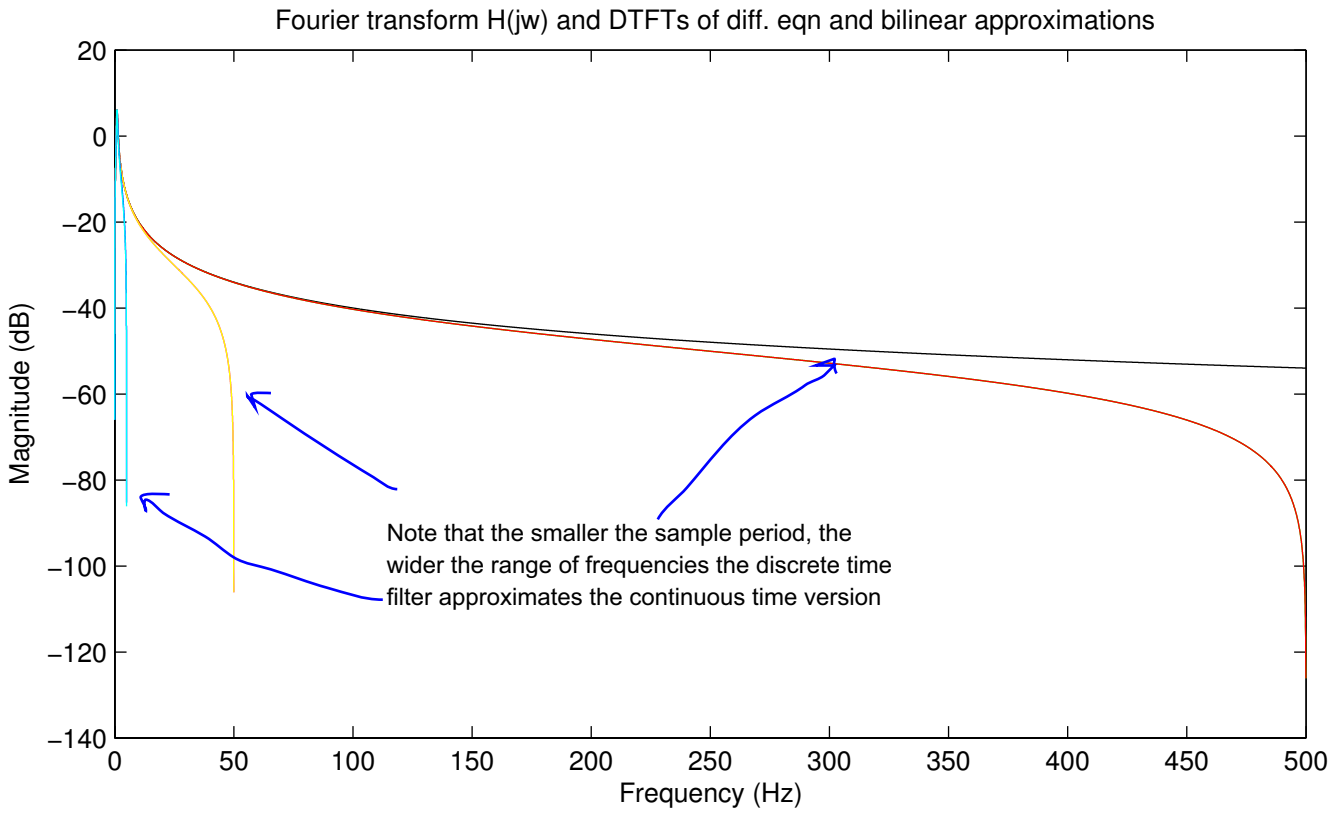

Phase (Degrees)
Frequency (Hz)
100
80
60
40
20
0
−20
−40
−60
−80
−100
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
H(s)
Diff. Eqn., .001*T_o
Bilinear, .001*T_o
Diff. Eqn., .01*T_o
Bilinear, .01*T_o
Diff. Eqn., .1*T_o
Bilinear, .1*T_o

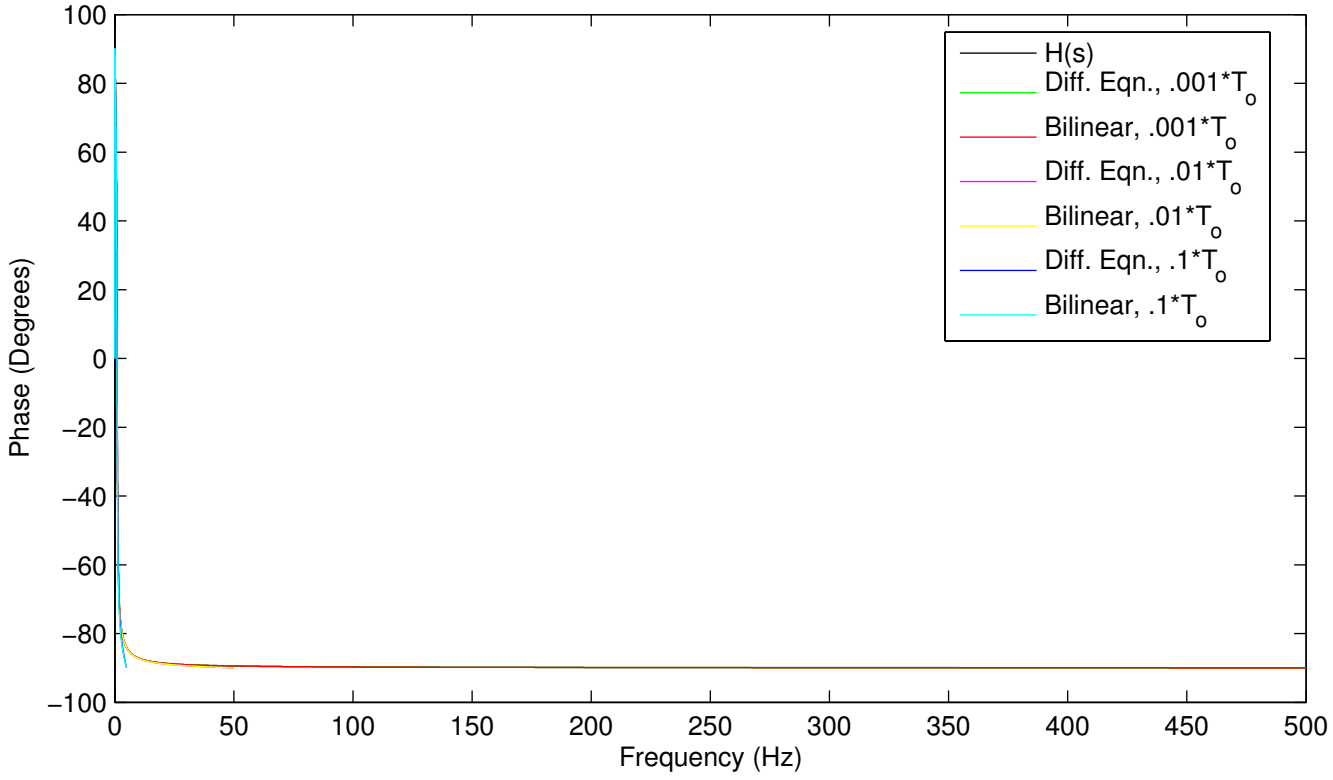
"Zoom IN" View

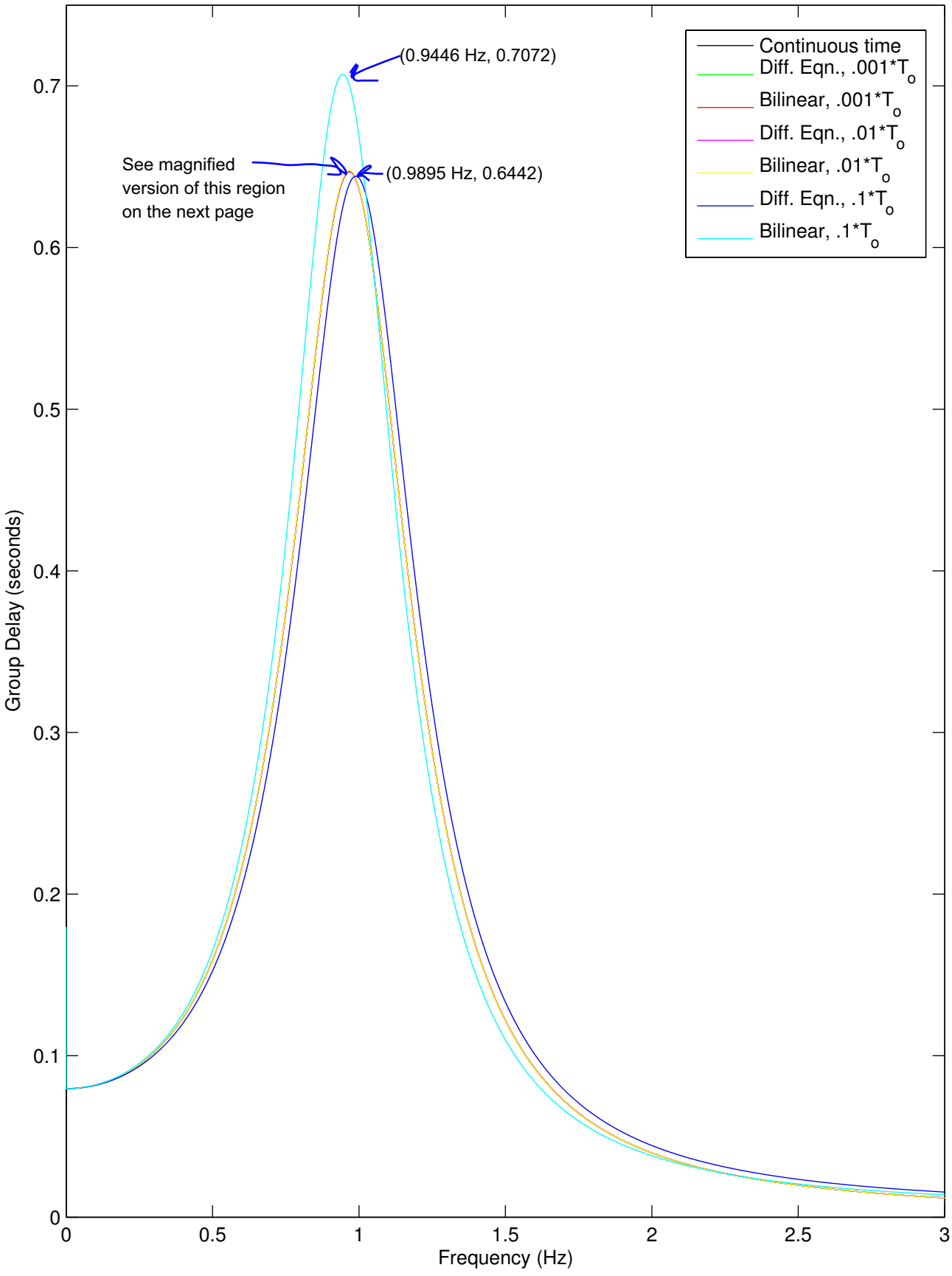

Fourier transform H(jw) and DTFTs of diff. eqn and bilinear approximations
Magnitude (dB)
Frequency (Hz)
Both the differential equation and bilinear mapping approximations with a sampling period of .001*To and .01*To mirror the continuous time transfer function very closely, with those with a period of .001*To coming the closest (more apparent if you zoom in at a closer scale).

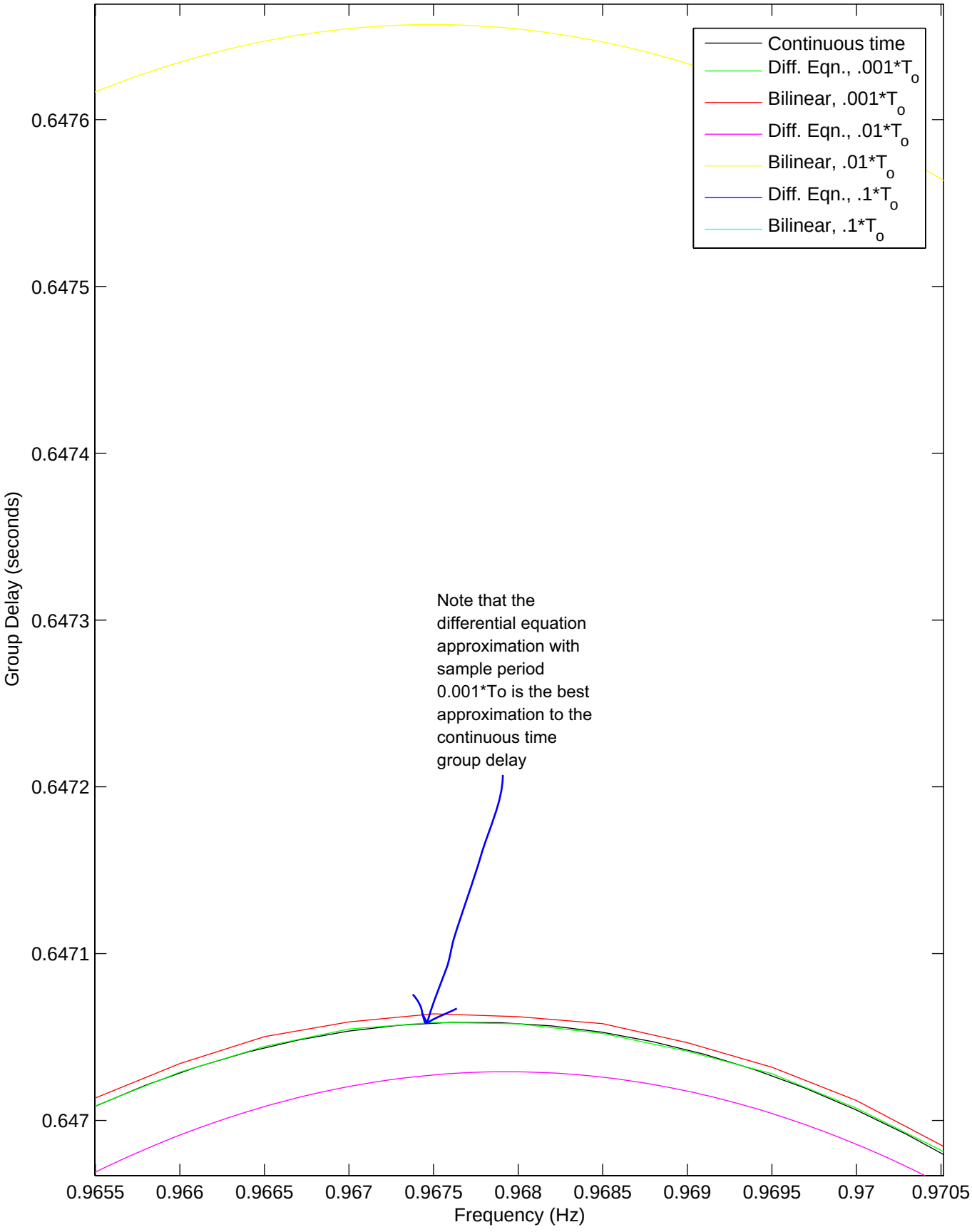

Phase (Degrees)
Frequency (Hz)
H(s)
Diff. Eqn., .001*T_o
Bilinear, .001*T_o
Diff. Eqn., .01*T_o
Bilinear, .01*T_o
Diff. Eqn., .1*T_o
Bilinear, .1*T_o

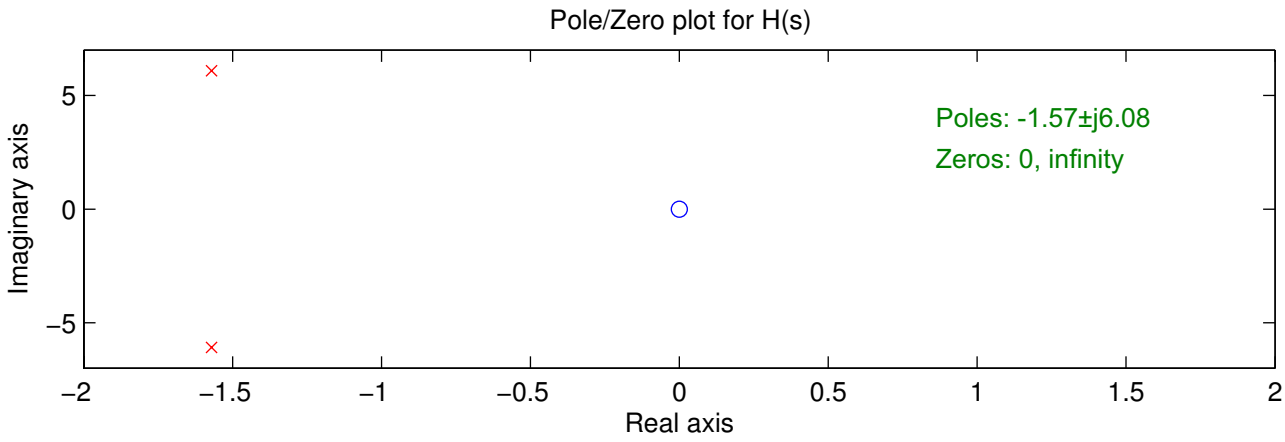
"Zoom OUT" View

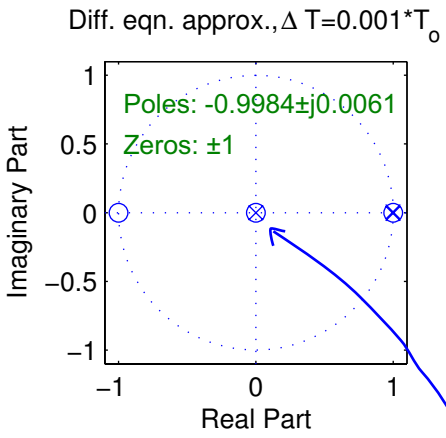


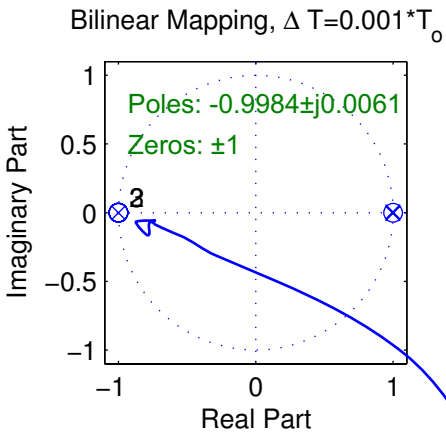


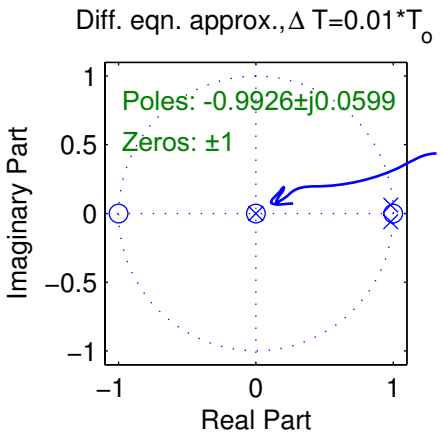

(0.9446 Hz, 0.7072)
See magnified
version of this region
on the next page
(0.9895 Hz, 0.6442)
Continuous time
Diff. Eqn., .001*T_o
Bilinear, .001*T_o
Diff. Eqn., .01*T_o
Bilinear, .01*T_o
Diff. Eqn., .1*T_o
Bilinear, .1*T_o
Group Delay (seconds)
Frequency (Hz)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0
0.5
1
1.5
2
2.5
3


Continuous time
Diff. Eqn., .001*T_o
Bilinear, .001*T_o
Diff. Eqn., .01*T_o
Bilinear, .01*T_o
Diff. Eqn., .1*T_o
Bilinear, .1*T_o
Note that the differential equation approximation with sample period 0.001*To is the best approximation to the continuous time group delay
Group Delay (seconds)
0.6476
0.6475
0.6474
0.6473
0.6472
0.6471
0.647
0.9655
0.966
0.9665
0.967
0.9675
0.968
0.9685
0.969
0.9695
0.97
0.9705
Frequency (Hz)

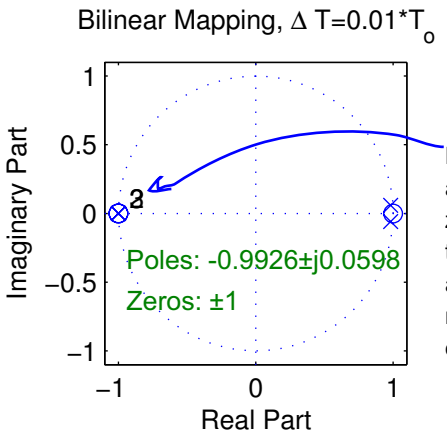




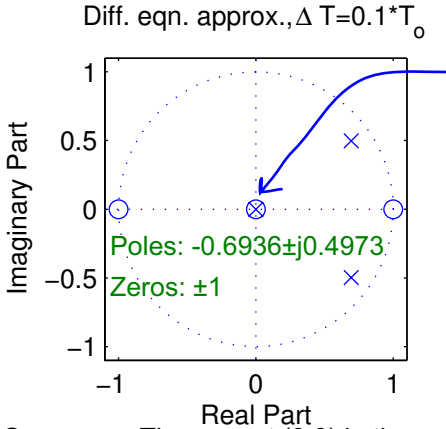


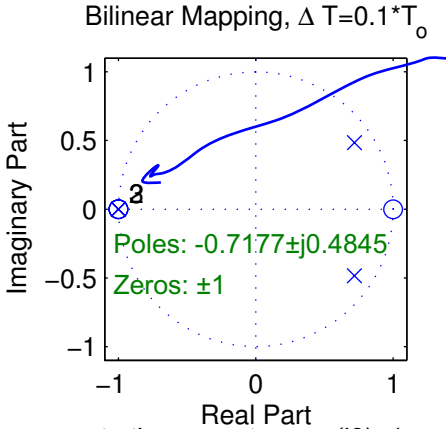


Note that the pole and zero at the origin on each of the diff. eqn. approx. cancel each other out.



Note that there is actually only one zero at (-1,0). Here there are 3 zeros and 2 poles, resulting in a net of one zero at (-1,0).





Summary: The zero at (0,0) in the continuous time version maps to the zero at z=exp(j0)=1 in the discrete time pole/zero plots. Since H(s) -> 0 as s->infinity, there is another zero at infinity. This zero maps to (-1,0) in the z plane. The two poles in H(s) at -1.57±j6.08 correspond to the two poles in each of the discrete transfer functions. The poles move due to the change in the sampling period, and their positions agree with the normalized frequencies shown in the transfer function plots.

```
%MATLAB code for Problem 4

clear all;
close all;

fo=1;           %Resonant Frequency
wo=2*pi*fo;
To=2*pi/wo;     %Resonant Period
d=.25;          %Damping Coefficient
dt=[.001 .01 .1];
DT=To*dt;
Tf=5;
NumFigures=8;

s = tf('s');
Hs = (s*wo)/(s^2 + 2*d*wo*s +wo^2);
figure(1);
subplot(4,2,1);
impulse(Hs,Tf)
title(['Impulse response of H(s) w/ f_o=',num2str(fo),', \delta=',num2str(d)])

figure(2);
pHs=pole(Hs);
zHs=zero(Hs);
subplot(4,1,1);
plot(real(pHs),imag(pHs),'xr',real(zHs),imag(zHs),'ob');
xlabel('Real axis');
ylabel('Imaginary axis');
title('Pole/Zero plot for H(s)');
axis([-2 2 -7 7]);

figure(4);
f=linspace(0,10,1e4);
s1=j*2*pi*f;
%H = freqs(wo,[1 2*d*wo wo^2],2*pi*f);
H = (s1*wo)./(s1.^2 + 2*d*wo*s1 +wo^2);
subplot(211);
plot(f,20*log10(abs(H)),'k');
hold on;
subplot(212);
plot(f,phase(H)*180/pi,'k');
hold on;

figure(3);
f2=linspace(0,3,10000);
omega=2*pi*f2;
group=((1/(d*wo))+(wo^2-omega.^2)./(2*d*wo*omega.^2))./(1+((wo^2-omega.^2).^2)./(4*(d^2)*↙
(wo^2)*(omega.^2)));
plot(f2,group,'k');

color=['g';'r';'m';'y';'b';'c'];
```

```
for k=1:3,
    z = tf('z',DT(k));
    a1=1/DT(k)+d*wo;
    a2=DT(k)*wo^2-2/DT(k);
    a3=1/DT(k)-d*wo;
    b=wo/2;

    figure(1);
    Hz=(b*(z-z^(-1)))/(a1*z+a2+a3*(z^(-1)));
    subplot(4,2,2*(k+1)-1);
    impulse(Hz,Tf)
    title(['Impulse Response w/ Diff. eqn. approx., \Delta T=',num2str(dt(k)),'*T_o']);
    axis([0 Tf -6.3*dt(k) 6.3*dt(k)]);

    s2=(2/DT(k))*((z-1)/(z+1));
    Hbilin = (s2*wo)/(s2^2 + 2*d*wo*s2 +wo^2);
    subplot(4,2,2*(k+1));
    impulse(Hbilin,Tf)
    title(['Impulse Response w/ Bilinear Mapping, \Delta T=',num2str(dt(k)),'*T_o']);
    axis([0 Tf -6.3*dt(k) 6.3*dt(k)]);

    figure(2);
    pHz=pole(Hz);
    zHz=zero(Hz);
    pHbilin=pole(Hbilin);
    zHbilin=zero(Hbilin);
    subplot(4,2,2*(k+1)-1);
    zplane(zHz,pHz)
    axis([-1.1 1.1 -1.1 1.1]);
    axis square;
    title(['Diff. eqn. approx., \Delta T=',num2str(dt(k)),'*T_o']);
    subplot(4,2,2*(k+1));
    zplane(zHbilin,pHbilin)
    axis([-1.1 1.1 -1.1 1.1]);
    axis square;
    title(['Bilinear Mapping, \Delta T=',num2str(dt(k)),'*T_o']);

    [bHz,aHz] = tfdata(Hz,'v');
    [bHbilin,aHbilin] = tfdata(Hbilin,'v');

    figure(4);
    subplot(212);
    [H_Hz,f_Hz]=freqz(bHz,aHz,1e4,1/DT(k));
    hold on;
    [H_bilin,f_bilin]=freqz(bHbilin,aHbilin,1e4,1/DT(k));
    subplot(211);
    plot(f_Hz,20*log10(abs(H_Hz)),color(2*(k)-1));
    hold on;
    plot(f_bilin,20*log10(abs(H_bilin)),color(2*(k)));
    hold on;
    xlabel('Frequency (Hz)');
    ylabel('Magnitude (dB)');
```

```
    subplot(212);
    plot(f_Hz,phase(H_Hz)*180/pi,color(2*(k)-1));
    hold on;
    plot(f_bilin,phase(H_bilin)*180/pi,color(2*(k)));
    hold on;
    xlabel('Frequency (Hz)');
    ylabel('Phase (Degrees)');

    figure(3);
    hold on;
    [bHz,aHz] = tfdata(Hz,'v');
    [Gd_Hz,F_Hz]=grpdelay(bHz,aHz,1e4,1/DT(k));
    plot(F_Hz,Gd_Hz*DT(k),color(2*(k)-1));
    hold on;
    [bHbilin,aHbilin] = tfdata(Hbilin,'v');
    [Gd_Hbilin,F_Hbilin]=grpdelay(bHbilin,aHbilin,1e4,1/DT(k));
    plot(F_Hbilin,Gd_Hbilin*DT(k),color(2*(k)));
end

figure(3);
legend('Continuous time','Diff. Eqn., .001*T_o', 'Bilinear, .001*T_o','Diff. Eqn., .↙
01*T_o', 'Bilinear, .01*T_o','Diff. Eqn., .1*T_o', 'Bilinear, .1*T_o');
 axis([0 3 0 .75])
ylabel('Group Delay (seconds)');
xlabel('Frequency (Hz)');

figure(4);
legend('H(s)','Diff. Eqn., .001*T_o', 'Bilinear, .001*T_o','Diff. Eqn., .01*T_o',↙
'Bilinear, .01*T_o','Diff. Eqn., .1*T_o', 'Bilinear, .1*T_o');
subplot(211);
title('Fourier transform H(jw) and DTFTs of diff. eqn and bilinear approximations');
axis([0 5 -70 7]);
subplot(212);
axis([0 5 -100 100]);
```

Solution Courtesy of Christ Richmond
from Solutions, Fall 2000

Problem 4 i) $w(\tau) = e^{-\frac{1}{2}(\frac{\tau}{M})^2} + e^{-\frac{1}{2}(\frac{\tau}{M})^2}(\frac{\tau}{T_H})^2$

$$\mathcal{F}_f\left[e^{-\frac{1}{2}(\frac{\tau}{M})^2}\right] = \mathcal{F}_f\left[e^{-\frac{1}{2M^2}\tau^2}\right]$$

$$= \sqrt{2\pi}\, M\, e^{-2\pi^2 M^2 f^2}$$

$$\frac{d}{d\tau} e^{-\frac{1}{2}(\frac{\tau}{M})^2} = -(\frac{\tau}{M^2}) e^{-\frac{1}{2}(\frac{\tau}{M})^2}$$

We know that $\mathcal{F}_f\left[\frac{d}{dt} f(t)\right] = (j2\pi f) F(f)$, so

$$\mathcal{F}_f\left[-(\frac{\tau}{M^2}) e^{-\frac{1}{2}(\frac{\tau}{M})^2}\right] = j(2\pi)^{\frac{3}{2}} M f\, e^{-2\pi^2 M^2 f^2}$$

We also know that

$$\mathcal{F}_f\left[t\, f(t)\right] = \frac{j}{2\pi}\frac{d}{df}[F(f)], \text{ so}$$

$$\mathcal{F}_f\left[-\frac{\tau^2}{M^2} e^{-\frac{1}{2}(\frac{\tau}{M})^2}\right] = -\sqrt{2\pi}\, M\left(e^{-2\pi^2 M^2 f^2} - 4\pi^2 M^2 f^2 e^{-2\pi^2 M^2 f^2}\right)$$

$$= -\sqrt{2\pi}\, M (1 - 4\pi^2 M^2 f^2) e^{-2\pi^2 M^2 f^2}$$

$$\Downarrow$$

$$\mathcal{F}_f\left[(\frac{\tau}{T_H})^2 e^{-\frac{1}{2}(\frac{\tau}{M})^2}\right] = \sqrt{2\pi}\,\frac{M^3}{T_H^2} e^{-2\pi^2 M^2 f^2}(1 - 4\pi^2 M^2 f^2)$$

So

$$\boxed{W(f) = \sqrt{2\pi}\, M e^{-2\pi^2 M^2 f^2}\left(1 + \frac{M^2}{T_H^2} - \frac{M^2}{T_H^2} 4\pi^2 M^2 f^2\right)}$$

We know that $E[\hat{S}_x(f|T)] = \int_{-\infty}^{+\infty} S_x(v)\, W(f-v)\, dv = S_x(f) * W(f)$

ii) let's observe $W(f)$:

$e^{-2\pi^2 M^2 f^2} \approx 3\times10^{-9}$ at $f = \frac{1}{M}$

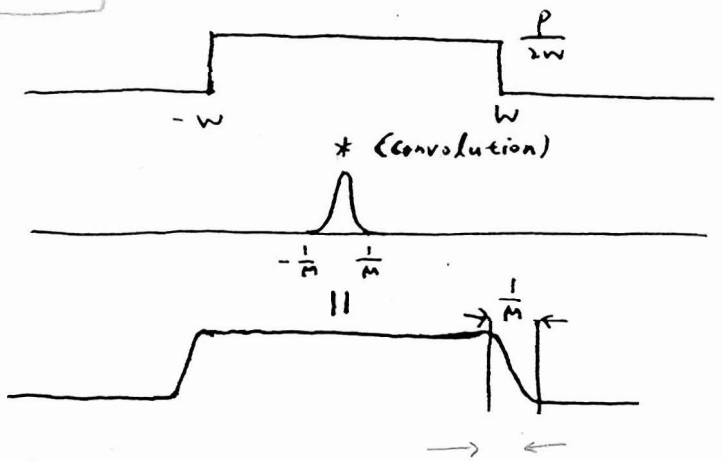
$(1 + \frac{M^2}{T_H^2} - \frac{M^2}{T_H^2} 4\pi^2 M^2 f^2)$ reaches 0 at $f_0 = \frac{1}{2\pi M}\sqrt{1+\frac{T_H^2}{M^2}} = \frac{\sqrt{M^2+T_H^2}}{2\pi M^2}$

$$f_0 \approx \begin{cases} \frac{T_H}{2\pi M^2} >> \frac{1}{M} & T_H >> M \\ \frac{1}{2\pi M} & T_H << M \end{cases}$$

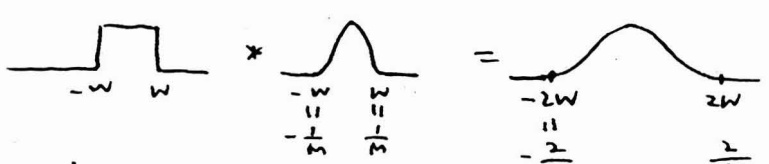
So we see that generally, the bandwidth of $W(f)$ is $\sim \frac{1}{M}$, in spite of the value of T_H.

Now let's see the resolution in estimating the boxcar spectrum:

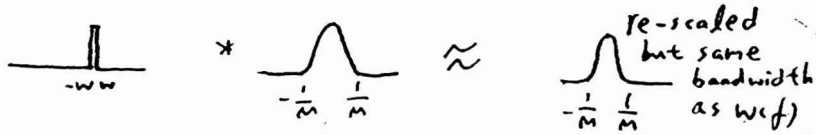
① $W >> \frac{1}{M}$, i.e., $WM >> 1$, resolution is $\frac{1}{M}$



② $W \approx \frac{1}{M}$, i.e. $WM \approx 1$,



③ $W << \frac{1}{M}$, i.e., $WM << 1$



There's no much point in talking about estimation resolution, because the output spectrum estimate is completely smeared out by $W(f)$.

iii) $\sigma_{\bar{x}}^2(f|T) = \frac{S_x^2(f)}{T} \int_{-\infty}^{+\infty} W^2(v)\, dv$

By Parseval's theorem

$$\int_{-\infty}^{+\infty} W^2(v)\, dv = \int_{-\infty}^{+\infty} w^2(\tau)\, d\tau$$

$$= \int_{-\infty}^{+\infty} e^{-\left(\frac{\tau}{M}\right)^2} \left[1 + 2\frac{\tau^2}{T_H^2} + \frac{\tau^4}{T_H^4}\right] d\tau$$

Note that $\int_{-\infty}^{+\infty} e^{-\frac{u^2}{2}} du = \sqrt{2\pi}$ $\int_{-\infty}^{+\infty} u^2 e^{-\frac{u^2}{2}} du = \sqrt{2\pi}$

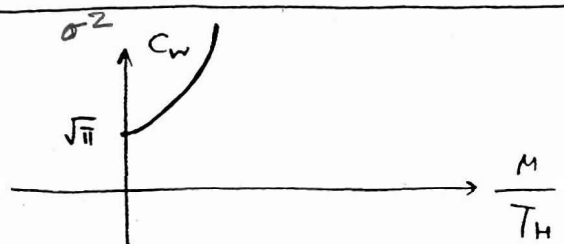
$\int_{-\infty}^{+\infty} u^4 e^{-\frac{u^2}{2}} du = 3\sqrt{2\pi}$, and then

$$\int_{-\infty}^{+\infty} e^{-\left(\frac{\tau}{M}\right)^2} d\tau = \sqrt{\pi}\, M$$

$$\int_{-\infty}^{+\infty} 2\frac{\tau^2}{T_H^2} e^{-\left(\frac{\tau}{M}\right)^2} d\tau = \frac{M^3}{T_H^2}\sqrt{\pi}$$

$$\int_{-\infty}^{+\infty} \frac{\tau^4}{T_H^4} e^{-\left(\frac{\tau}{M}\right)^2} d\tau = \frac{3}{4}\frac{M^5}{T_H^4}\sqrt{\pi}$$

$$\Rightarrow \int_{-\infty}^{+\infty} W^2(v)\, dv = \int_{-\infty}^{+\infty} w^2(\tau)\, d\tau = \sqrt{\pi}\, M \left[1 + \left(\frac{M}{T_H}\right)^2 + \frac{3}{4}\left(\frac{M}{T_H}\right)^4\right]$$

$$\Rightarrow \frac{\sigma_{\bar{x}}^2(f|T)}{S_x^2(f)} = C_w \frac{M}{T}$$

where $\boxed{C_w = \sqrt{\pi}\left[1 + \left(\frac{M}{T_H}\right)^2 + \frac{3}{4}\left(\frac{M}{T_H}\right)^4\right]}$