Quiz #1 solutions

Problem 1

i) $$X(f) = \begin{cases} jX_0 & f \in [f_0, f_0+W] \\ -jX_0 & f \in [-f_0-W, -f_0] \end{cases}$$

Note that conjugate symmetry holds $\left(X(f) = X^*(-f) \right)$

Obviously, $\tilde{X}(f) = jX_0$, $f \in [0, W]$ (carrier freq $= f_0$)

$$X_c(f) \triangleq \text{Even}\left[\tilde{X}(f)\right] = \frac{1}{2}\left(\tilde{X}(f) + \tilde{X}^*(-f) \right) = \begin{cases} j\frac{X_0}{2}, & f \in [0, W] \\ -j\frac{X_0}{2}, & f \in [-W, 0] \end{cases}$$

$$X_s(f) \triangleq j\,\text{Odd}\,\tilde{X}(f) = \frac{j}{2}\left(\tilde{X}(f) - \tilde{X}^*(-f) \right) = \begin{cases} -X_0/2, & f \in [0, W] \\ -X_0/2, & f \in [-W, 0] \end{cases}$$

ii) $$x_c(t) \triangleq \text{Re}\left[\tilde{x}(t)\right] = \text{Re}\left(\int_0^W jX_0 e^{j2\pi f t}\, df \right] =$$

$$= \text{Re}\left[jX_0 \frac{e^{j2\pi W t} - 1}{j2\pi t} \right] = \text{Re}\left[X_0 e^{j2\pi \frac{W}{2} t} \frac{\sin(\pi W t)}{\pi t} \right]$$

$$= X_0 \,\text{sinc}(\pi W t)\, \text{Re}\left[e^{j2\pi \frac{W}{2} t} \right] = X_0 \,\text{sinc}(\pi W t) \cos\left(2\pi \frac{W}{2} t\right)$$

$$x_s(t) \triangleq -\text{Im}\left[\tilde{x}(t)\right] = -\text{Im}\left[\int_0^W jX_0 e^{j2\pi f t}\, df \right]$$

$$= -\text{Im}\left[X_0 e^{j2\pi \frac{W}{2} t} \,\text{sinc}(\pi W t) \right] = -X_0 \,\text{sinc}(\pi W t) \sin\left(2\pi \frac{W}{2} t\right)$$

iii) $X_1(f) = \begin{cases} jX_0, & f \in [f_0+\delta f,\ f_0+W+\delta f] \\ -jX_0, & f \in [-f_0-W-\delta f,\ -f_0-\delta f] \end{cases}$

thus $\tilde{X}_1(f) = jX_0 \quad f \in [\delta f, \delta f + W]$ or

$\tilde{X}_1(f) = \tilde{X}(f-\delta f)$

$X_{1_E}(f) = \text{Even}[\tilde{X}_1(f)] = \begin{cases} jX_0/2, & f \in [\delta f, W+\delta f] \\ -jX_0/2, & f \in [-W-\delta f, -\delta f] \end{cases}$

$X_{1_S}(f) = j\,\text{odd}[\tilde{X}_1(f)] = \begin{cases} -X_0/2, & f \in [\delta f, W+\delta f] \\ -X_0/2, & f \in [-W-\delta f, -\delta f] \end{cases}$

$x_{1_c}(t) = \text{Re}[\tilde{x}_1(t)] = \text{Re}\left[\int_{\delta f}^{\delta f+W} jX_0 e^{j2\pi ft} df\right] =$

$= \text{Re}\left[e^{+j2\pi\delta ft}\int_0^W jX_0 e^{j2\pi ft} df\right] = \text{Re}\left[e^{+j2\pi\delta ft} X_0 e^{j2\pi\frac{W}{2}t} \text{sinc}(\pi Wt)\right]$

$= X_0 \,\text{sinc}(\pi Wt) \cos\left(2\pi\left(\frac{W}{2}+\delta f\right)t\right)$

similarly, $x_{1_s}(t) = -X_0 \,\text{sinc}(\pi Wt) \sin\left(2\pi\left(\frac{W}{2}+\delta f\right)t\right)$

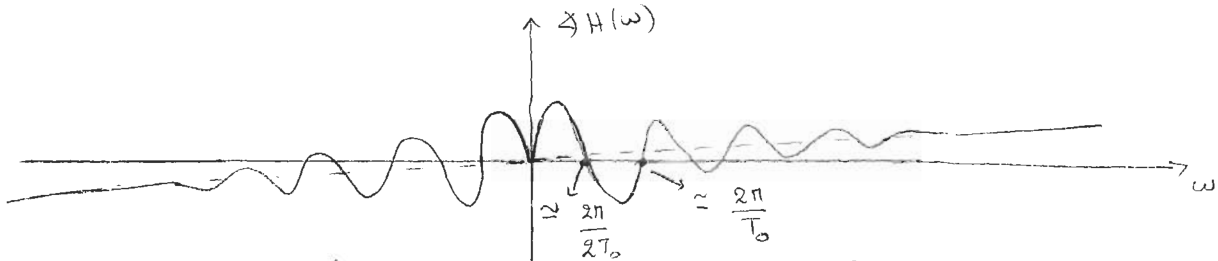
iv) if X_0 is complex then from part ii)

$x_c(t) = \text{Re}\left[X_0 e^{j2\pi\frac{W}{2}t} \text{sinc}(\pi Wt)\right] = \text{sinc}(\pi Wt)\, \text{Re}\left[X_0 e^{j2\pi\frac{W}{2}t}\right]$

$x_s(t) = -\text{sinc}(\pi Wt)\, \text{Im}\left[X_0 e^{j2\pi\frac{W}{2}t}\right]$

Problem 2

i) $$\measuredangle H(\omega) = \left(S_0 \omega + \alpha \sin(|\omega T_0|) e^{-|\omega T_1|} \right) R$$



the contribution of the first nulls comes from $\sin|\omega T_0|$ only because $T_0 \gg T_1$ (exp. decay does not affect them) and $\alpha \gg S_0 \omega$

ii) $$T_p(\omega) \triangleq -\frac{1}{\omega} \measuredangle H(\omega) = -R\left(S_0 + \alpha \sin(|\omega T_0|) \frac{e^{-|\omega T_1|}}{\omega} \right)$$

I need to know $$\frac{d}{dx}|x| = \frac{d}{dx}\sqrt{x^2} = \frac{1}{2\sqrt{x^2}} 2x = \frac{x}{|x|}$$

$$= \operatorname{sgn}(x)\frac{|x|}{|x|} = \operatorname{sgn}(x) = \begin{cases} 1, & x>0 \\ -1, & x<0 \end{cases}$$

hence, $$T_g(\omega) \triangleq -\frac{d}{d\omega} \measuredangle H(\omega) =$$

$$= -R\left[S_0 + \alpha\left(\cos|\omega T_0| \operatorname{sgn}(\omega T_0) T_0 e^{-|\omega T_1|} + \sin|\omega T_0| e^{-|\omega T_1|} T_1(-\operatorname{sgn}(\omega T_1)) \right)\right]$$

$$= -R\left[S_0 + \alpha e^{-|\omega T_1|} \left(T_0 \operatorname{sgn}(\omega T_0) \cos(|\omega T_0|) - T_1 \operatorname{sgn}(\omega T_1) \sin(|\omega T_0|) \right) \right]$$

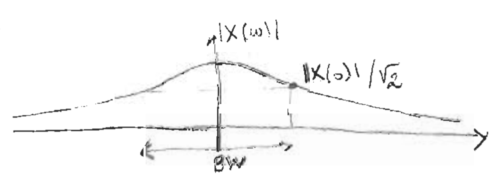
for $\omega > 0$ $$T_g(\omega) = -R\left[S_0 + \alpha e^{-\omega T_1} \left(T_0 \cos(\omega T_0) - T_1 \sin(\omega T_0) \right) \right]$$

iii) $x(t) = X_0 e^{-2t/T_0} u(t)$ input

$$X(f) = \int_0^\infty X_0 e^{-2t/T_0} e^{-j2\pi f t} dt = X_0 \int_0^\infty e^{-j2\pi t\left(f + \frac{2}{T_0 2\pi j}\right)} dt$$

$$X_0 \left(0 - \frac{1}{-j2\pi\left(f + \frac{2}{T_0 2\pi j}\right)} \right) = \frac{X_0}{j2\pi f + \frac{2}{T_0}} \Rightarrow X(\omega) = \frac{X_0}{j\omega + \frac{2}{T_0}}$$

$$\Rightarrow |X(\omega)| = \frac{|X_0|}{\sqrt{\omega^2 + 4/T_0^2}}$$



Define BW where the magnitude of $X(\omega)$ falls to $\frac{|X(0)|}{\sqrt{2}}$. Hence $\frac{|X(0)|}{\sqrt{2}} = \frac{|X_0|}{\sqrt{\omega^2 + 4/T_0^2}} \Rightarrow$

$$\frac{|X_0|}{\sqrt{2}\,\frac{2}{T_0}} = \frac{|X_0|}{\sqrt{\omega^2 + 4/T_0^2}} \Rightarrow \omega^2 + \frac{4}{T_0^2} = 2\,\frac{4}{T_0^2} \Rightarrow$$

$$\omega = \pm \frac{2}{T_0}$$ thus BW of $x(t)$ is $\pm \frac{2}{T_0}$.

The output of the filter is:

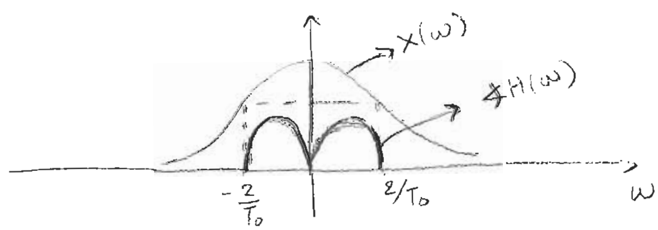
$$Y(\omega) = X(\omega)\,H(\omega) = X(\omega)\,|H(\omega)|\,e^{j\measuredangle H(\omega)} =$$

$$= \frac{X_0}{j\omega + \frac{2}{T_0}}\,|H(\omega)|\,e^{jR\left(S_0\omega + \alpha \sin|\omega T_0|\,e^{-|\omega T_1|}\right)}$$

The region of interest is when $\omega \in \left[-\frac{2}{T_0}, \frac{2}{T_0}\right]$ i.e. the "effective" BW of $x(t)$.

In this region the contribution of $\measuredangle H(\omega)$ comes from $\alpha \sin|\omega T_0|$ only so we can approximate

$$\measuredangle H(\omega) \simeq \alpha R \sin|\omega T_0|.$$



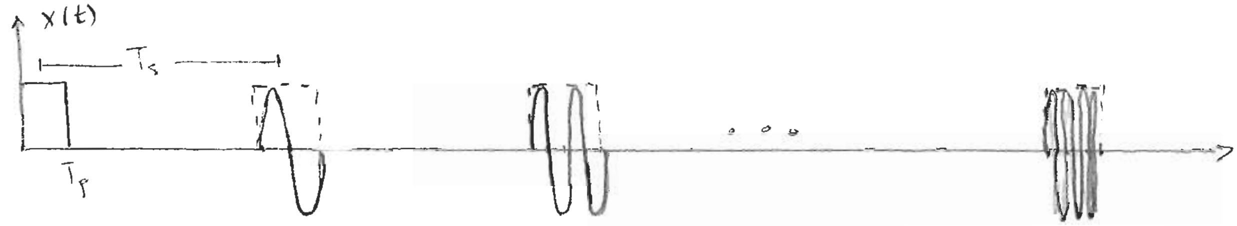
Obviously dispersion cannot be neglected so

$$Y(\omega) \simeq \frac{X_0}{j\omega + \frac{2}{T_0}}\,H_0\,e^{jR\alpha \sin|\omega T_0|}$$

and $$y(t) \simeq \int_{-2/T_0}^{2/T_0} \frac{X_0 H_0}{j\omega + 2/T_0}\,e^{jR\alpha \sin|\omega T_0|}\,e^{j\omega t}\,\frac{d\omega}{2\pi}$$

Problem 3

i) The transmitted signal is:



We want $\int x^2(t)\,dt = 1 \Rightarrow$

$$\sum_{n=1}^{N} \int_{(n-1)T_s}^{(n-1)T_s+T_p} A^2 \sin^2\left(\omega_n\left(t-(n-1)T_s\right)\right) dt = 1 \Rightarrow$$

$$\sum_{n=1}^{N} \int_0^{T_p} A^2 \sin^2(\omega_n t)\,dt = 1 \Rightarrow \sum_{n=1}^{N} \int_0^{T_p} A^2 \left(\frac{1-\cos(2\omega_n t)}{2}\right) dt = 1$$

$$\frac{A^2}{2} \sum_{n=1}^{N} \int_0^{T_p} dt - \underbrace{\int_0^{T_p} \cos\left(2\cdot 2\pi (n-1)\frac{1}{T_p} t\right) dt}_{0} = 1 \Rightarrow$$

$$\frac{A^2 N T_p}{2} = 1 \Rightarrow A = \left(\frac{2}{NT_p}\right)^{1/2}.$$

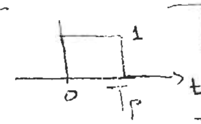
ii) $$\phi(0,\Delta f) = \int_{-\infty}^{\infty} x^2(t)\, e^{-j2\pi\Delta f t}\,dt =$$

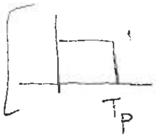
$$= \sum_{n=1}^{N} \int_{(n-1)T_s}^{(n-1)T_s+T_p} A^2 \sin^2\left(\omega_n\left(t-(n-1)T_s\right)\right) e^{-j2\pi\Delta f t}\,dt =$$

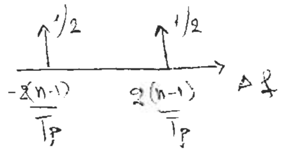
$$= \sum_{n=1}^{N} \int_0^{T_p} A^2 \sin^2(\omega_n t)\, e^{-j2\pi\Delta f\left(t+(n-1)T_s\right)}\,dt =$$

$$= \sum_{n=1}^{N} e^{-j2\pi\Delta f(n-1)T_s} \int_0^{T_p} \frac{A^2}{2}\left(1-\cos(2\omega_n t)\right) e^{-j2\pi\Delta f t}\,dt$$

$$= \sum_{n=1}^{N} \frac{A^2}{2} e^{-j2\pi \Delta f (n-1)T_s} \left[\int_0^{T_p} 1 \cdot e^{-j2\pi\Delta f t}\, dt - \int_0^{T_p} \cos(2\omega_n t) e^{-j2\pi\Delta f t}\, dt \right]$$

$$\int_0^{T_p} e^{-j2\pi\Delta f t}\, dt = F\Bigg[$$  $$\Bigg] = T_p e^{-j2\pi\Delta f T_p/2} \operatorname{sinc}(\pi T_p \Delta f)$$

$$\int_0^{T_p} \cos(2\omega_n t) e^{-j2\pi\Delta f t}\, dt = F\Bigg[$$  $$\cdot \cos(2\omega_n t)\Bigg]$$

$$= T_p e^{-j2\pi\Delta f T_p/2} \operatorname{sinc}(\pi T_p \Delta f) *$$ 

$$= \frac{T_p}{2} e^{-j2\pi\left(\Delta f \pm (n-1)\frac{2}{T_p}\right)\frac{T_p}{2}} \operatorname{sinc}\left(\pi T_p \left(\Delta f \pm \frac{2(n-1)}{T_p}\right)\right)$$

Hence the end result is:

$$\phi(0,\Delta f) = A^2 \frac{1}{2} \sum_{n=1}^{N} e^{-j2\pi\Delta f(n-1)T_s} \Bigg[T_p e^{-j2\pi\Delta f T_p/2} \operatorname{sinc}(\pi T_p \Delta f) -$$

$$\frac{T_p}{2} e^{-j2\pi\left(\Delta f + (n-1)\frac{2}{T_p}\right)\frac{T_p}{2}} \operatorname{sinc}\left(\pi T_p \left(\Delta f + \frac{2(n-1)}{T_p}\right)\right) -$$

$$\frac{T_p}{2} e^{-j2\pi\left(\Delta f - (n-1)\frac{2}{T_p}\right)\frac{T_p}{2}} \operatorname{sinc}\left(\pi T_p \left(\Delta f - \frac{2(n-1)}{T_p}\right)\right) \Bigg]$$

For Doppler accuracy the shifted $\operatorname{sinc}\left(\pi T_p\left(\Delta f - \frac{2(n-1)}{T_p}\right)\right)$ do not contribute so we approximate.

$$\phi(0,\Delta f) \simeq \frac{A^2}{2} \sum_{n=1}^{N} \left[e^{-j2\pi\Delta f(n-1)T_s} T_p e^{-j2\pi\Delta f T_p/2} \operatorname{sinc}(\pi T_p \Delta f) \right]$$

$$- T_p e^{-j2\pi\Delta f T_p/2} \operatorname{sinc}(\pi T_p \Delta f) =$$

$$= A^2 \frac{T_p}{2} \sum_{n=2}^{N} e^{-j2\pi\Delta f(n-1)T_s} \cdot e^{-j2\pi\Delta f T_p/2} \operatorname{sinc}(\pi T_p \Delta f)$$

$$= A^2 \frac{T_p}{2} \operatorname{sinc}(\pi T_p \Delta f)\, e^{-j\pi \Delta f T_p} \underbrace{\sum_{n=2}^{N} e^{-j2\pi \Delta f (n-1) T_s}}_{\text{sum of finite geometric series}}$$

$$\sum_{k=1}^{n} r^k = \frac{r(1-r^n)}{1-r}$$

$$= A^2 \frac{T_p}{2} \operatorname{sinc}(\pi T_p \Delta f)\, e^{-j\pi \Delta f T_p}\, e^{-j2\pi \Delta f T_s} \left(\frac{1 - e^{-j2\pi \Delta f (N-1) T_s}}{1 - e^{-j2\pi \Delta f T_s}} \right)$$

$$= A^2 \frac{T_p}{2} e^{-j\pi \Delta f (T_p + T_s)} \operatorname{sinc}(\pi T_p \Delta f) \frac{\sin(\pi (N-1) T_s \Delta f)}{\sin(\pi T_s \Delta f)} \frac{e^{-j\pi \Delta f (n-1) T_s}}{e^{-j\pi \Delta f T_s}}$$

So $\phi(0, \Delta f)$ has peaks of width $\frac{1}{(N-1)T_s}$ and is periodic at $\Delta f = \frac{1}{T_s}$. This implies that the Doppler resolution is determined by the signal duration but there are ambiguous sidelobes at $\frac{1}{T_s}$.

Observing the structure of $\phi(0, \Delta f)$ there exist additional sidelobes @ $\frac{n}{T_p}$ $n = 1 \ldots N$.

$$\phi(\Delta T, 0) = \int_{-\infty}^{\infty} x\left(t - \frac{\Delta T}{2}\right) x\left(t + \frac{\Delta T}{2}\right) dt = \int_{-\infty}^{\infty} x(t)\, x(t - \Delta T)\, dt$$

when $\boxed{0 \leq \Delta T \leq T_p}$ we are looking at the mainlobe width

$$\phi(\Delta T, 0) = \sum_{n=1}^{N} \int_{(n-1)T_s + \Delta T}^{(n-1)T_s + T_p} A \sin\left(\omega_n (t - (n-1)T_s)\right) A \sin\left(\omega_n (t - (n-1)T_s - \Delta T)\right) dt$$

using $\sin A \sin B = \frac{1}{2}\left(\cos(A-B) - \cos(A+B)\right)$

$$\phi(\Delta T, 0) = \sum_{n=1}^{N} \frac{A^2}{2} \int_{(n-1)T_s + \Delta T}^{(n-1)T_s + T_p} \cos(\omega_n \Delta T) - \cos\left(2\omega_n (t - (n-1)T_s) - \omega_n \Delta T\right) dt$$

$$= \sum_{n=1}^{N} \frac{A^2}{2} \left[\cos(\omega_n \Delta T)(T_p - \Delta T) - \int_{\Delta T}^{T_p} \cos(2\omega_n t - \omega_n \Delta T)\, dt \right]$$

$$= \sum_{n=1}^{N} \frac{A^2}{2} \left[(T_p - \Delta T) \cos(\omega_n \Delta T) - \frac{\sin \omega_n (2T_p - \Delta T) - \sin(\omega_n \Delta T)}{2\omega_n} \right]$$

Observe that the first term of the sum defines the range resolution which is $\Delta T = T_p$ (first zero). Hence range resolution $\sim T_p$.

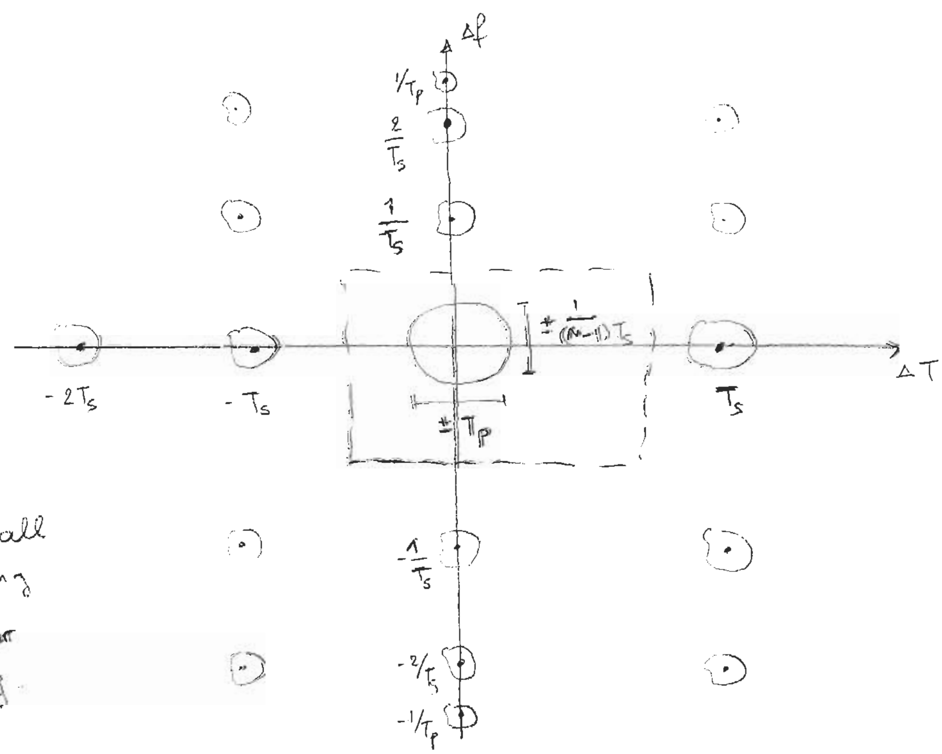
When $T_p < \Delta T < T_s - T_p$ then there is no overlap so $\phi(\Delta T, 0) = 0$.

When $T_s \leq \Delta T \leq T_s + T_p$ then we are looking at the width of the first sidelobe

$$\phi(\Delta T, 0) = \sum_{n=2}^{N} \int_{(n-2)T_s + \Delta T}^{(n-1)T_s + T_p} A \sin \omega_n (t - (n-1)T_s) \; A \sin \omega_{n-1} (t - (n-2)T_s - \Delta T) \; dt$$

$$= \sum_{n=2}^{N} \frac{A^2}{2} \int_{(n-2)T_s + \Delta T}^{(n-2)T_s + T_p} \cos\left[\omega_n (t - (n-1)T_s) - \omega_{n-1}\left(t - (n-2)T_s - \Delta T\right)\right] - \cos\left[\omega_n\left(t - (n-1)T_s\right) + \omega_{n-1}\left(t - (n-2)T_s - \Delta T\right)\right] dt$$

due to this constructive and distructive summation of cosines the sidelobe around T_s is severely smaller than the mainlobe.
So how does $\phi(\Delta T, \Delta f)$ look like?



make T_p small for range gating
Increase N for Doppler gating.