Pset #1 - solutions

Problem 1

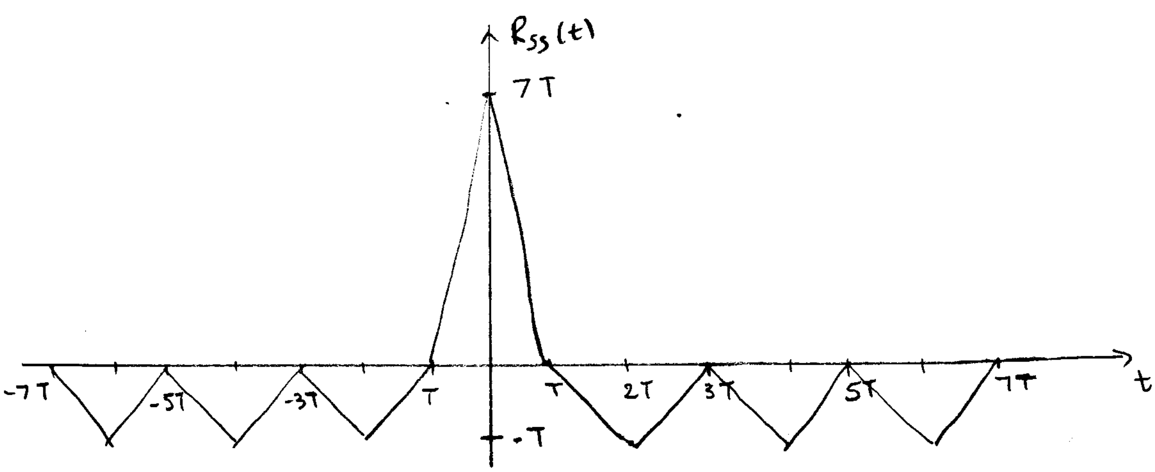
a) let $T_d = 0$, $r(t) = s(t)$ where $s(t)$ is given in Figure 1(a).
The we have the following block diagram

$$r(t) = s(t) \longrightarrow \boxed{s(-t) = h(t)} \longrightarrow y(t)$$

thus, $y(t) = s(t) * h(t) = \int_{-\infty}^{\infty} s(\tau)\, h(t-\tau)\, d\tau$

$$= \int_{-\infty}^{\infty} s(\tau)\, s(\tau - t)\, d\tau = R_{ss}(t)$$

So the output is the autocorrelation of $s(t)$.
The resulting $y(t)$ is shown below



If T_d is non zero then $r(t) = s(t - T_d)$ and
$y(t) = s(t - T_d) * s(-t) = \int_{-\infty}^{\infty} s(\tau - T_d)\, s(\tau - t)\, d\tau$

$$\overset{\tau - T_d = v}{=} \int_{-\infty}^{\infty} s(v)\, s(v - (t - T_d))\, dv = R_{ss}(t - T_d)$$

So the output is time-shifted by T_d as we should expect. The max will appear at $t = T_d$.

Therefore an algorithm estimating the range of a target could be the following:

1) find time corresponding to the max value of the output of the filter which is T_d (2-way travel time)
2) range is : $R = \frac{c \cdot T_d}{2}$ (1-way travel time)

where c is the speed in the propagation medium.

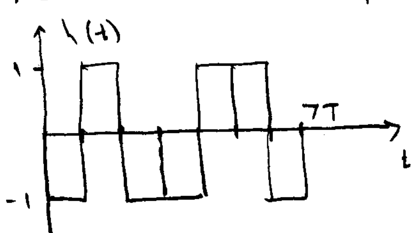
b) If $h(t) = s(\underbrace{7T}_{delay} - t)$ then the output becomes

$$y(t) = s(t - T_d) * s(-(t - 7T)) = \int_{-\infty}^{\infty} s(\tau - T_d)\, s(-\tau + 7T - t)\, d\tau$$

$$\overset{v = \tau - T_d}{=} \int_{-\infty}^{\infty} s(v)\, s(v - (t - T_d + 7T))\, dt = R_{ss}(t - (7T + T_d))$$

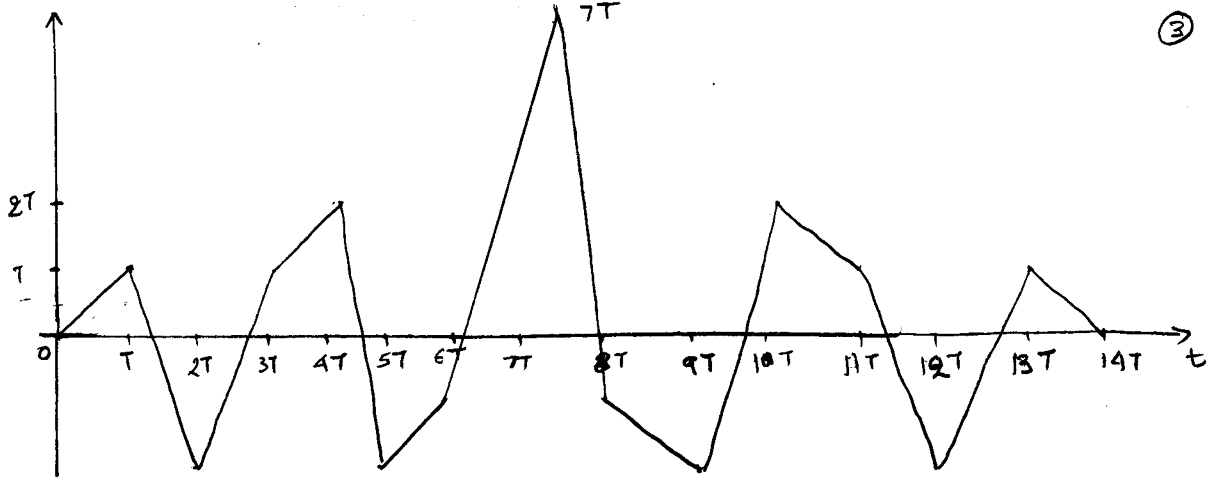
so the output is time shifted by $7T + T_d$ as expected. The max value is at $7T + T_d$ so the two-way travel time is $T_d - 7T$ and the range is $R = c \cdot \frac{t_m}{2} = \underline{\frac{c \cdot (T_d - 7T)}{2}}$.

c) the matched filter impulse responce for this input is



$h(t) = s(7T - t)$

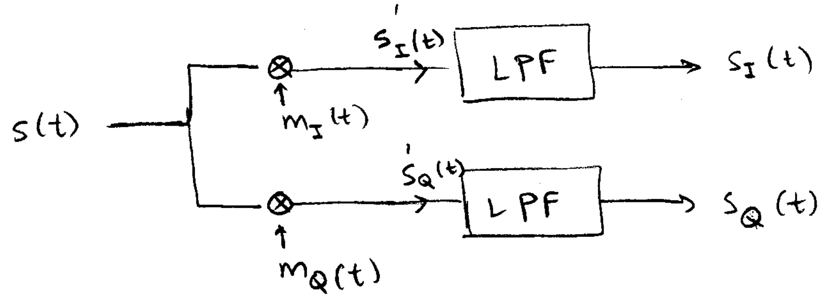
the output $y(t) = \int_{-\infty}^{\infty} s(\tau)\, s(\tau - (t - 7T))$ is



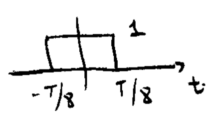
Obviously if we reverse one bit in the Barker code the ouput of the filter has bigger sidelobes hence we are more vulnerable in making a mistake in high noise conditions.

Problem 2

a) From the given scheme we realize that the incoming narrowband signal $s(t)$ is being demodulated by a train of pulses.



It is easier if we do the analysis in the frequency domain. $m_I(t)$ is a train of pulses so it can be written as:

$m_I(t) =$ 

$\ast \sum_{n=-\infty}^{\infty} \delta(t - nT)$.

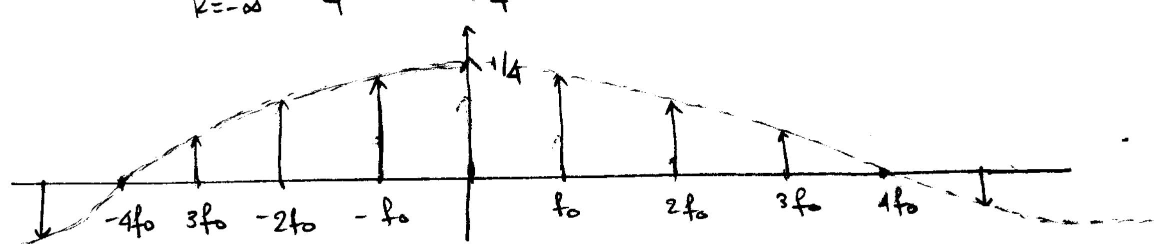
The F.T. of $m_I(t)$ is then:

$$M_I(f) = \frac{T}{4} \cdot \operatorname{sinc}\left(\pi f \frac{T}{4}\right) \cdot \frac{1}{T} \sum_{k=-\infty}^{\infty} \delta\left(f - \frac{k}{T}\right) =$$

$$= \frac{1}{4} \sum_{k=-\infty}^{\infty} \operatorname{sinc}\left(\pi f \frac{T}{4}\right) \delta\left(f - \frac{k}{T}\right) =$$

$$= \frac{1}{4} \sum_{k=-\infty}^{\infty} \operatorname{sinc}\left(\pi \frac{k}{\cancel{T}} \frac{\cancel{T}}{4}\right) \delta\left(f - \frac{k}{T}\right) =$$

$$\overset{T=\frac{1}{f_0}}{=} \sum_{k=-\infty}^{\infty} \frac{1}{4} \operatorname{sinc}\left(\frac{\pi k}{4}\right) \delta(f - k f_0)$$



Hence $\quad s_I'(t) = s(t)\cdot m_I(t) \Rightarrow$

$$S_I'(f) = S(f) * M_I(f) = S(f) * \sum_{k=-\infty}^{\infty} \frac{1}{4}\,\mathrm{sinc}\left(\frac{k\pi}{4}\right)\delta(f - kf_0)$$

$$= \sum_{k=-\infty}^{\infty} \underbrace{\frac{1}{4}\,\mathrm{sinc}\left(\frac{k\pi}{4}\right)}_{\text{weights}} S(f - kf_0)$$

Thus the spectrum $S_I'(f)$ is a weighted superposition of shifted spectrums $S(f \pm kf_0)$, $k = 0, \pm 1, \pm 2, \ldots$

Substituting $\quad S(f) = S_0(f - f_0) + S_0^*(-(f+f_0))\quad$ we have

$$S_I'(f) = \sum_{k=-\infty}^{\infty} \frac{1}{4}\,\mathrm{sinc}\left(\frac{k\pi}{4}\right)\left[S_0\left(f - (k+1)f_0\right) + S_0^*\left(-\left(f - (k-1)f_0\right)\right)\right]$$

After lowpass filtering only the baseband components survive i.e. for $k = 1$ only $S_0^*(-(f - (k-1)f_0))$ survives
for $k = -1$ only $S_0(f - (k+1)f_0)$ survives

Thus, $\quad S_I(f) = \frac{1}{4}\,\mathrm{sinc}\left(\frac{k\pi}{4}\right)\left(S_0(f) + S_0^*(-f)\right) =$

$$= \frac{\sqrt{2}}{\pi}\cdot \underline{\mathrm{Even}\left(S_0(f)\right)}$$

Similarly

$$m_Q(t) = m_I\left(t - \frac{T}{4}\right) \Rightarrow M_Q(f) = M_I(f)\, e^{-j2\pi f\frac{T}{4}} =$$

$$= \sum_{k=-\infty}^{\infty} \frac{1}{4}\,\mathrm{sinc}\left(\frac{\pi k}{4}\right)\delta(f - kf_0)\, e^{-j2\pi f\frac{T}{4}} =$$

$$= \sum_{k=-\infty}^{\infty} \frac{1}{4}\,\mathrm{sinc}\left(\frac{\pi k}{4}\right) e^{-j2\pi k f_0 \frac{T}{4}}\,\delta(f - kf_0)$$

$$= \sum_{k=-\infty}^{\infty} \frac{1}{4}\,\mathrm{sinc}\left(\frac{\pi k}{4}\right) \underbrace{e^{-jk\frac{\pi}{2}}}_{j^{-k}}\,\delta(f - kf_0) = \sum_{k=-\infty}^{\infty} j^{-k}\,\frac{1}{4}\,\mathrm{sinc}\left(\frac{\pi k}{4}\right)\delta(f - kf_0)$$

Hence, $S_Q'(f) = \sum_{k=-\infty}^{\infty} j^{-k} \frac{1}{4}\,\text{sinc}\left(\frac{k\pi}{4}\right) S_0\left(f-(k+1)f_0\right) + S_0^*\left(-\left(f-(k-1)f_0\right)\right)$

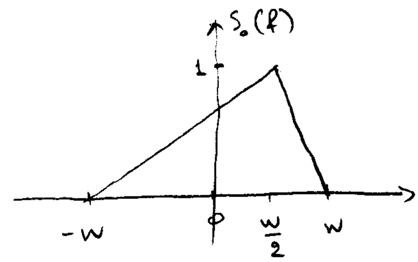
for the same reason as before only $S_0(f)$ $(k=-1)$ and $S_0^*(-f)$ $(k=1)$ survive so:

$$S_Q(f) = j\,\frac{1}{4}\,\text{sinc}\left(\frac{k\pi}{4}\right) S_0(f) - j\,\frac{1}{4}\,\text{sinc}\left(\frac{k\pi}{4}\right) S_0^*(-f)$$

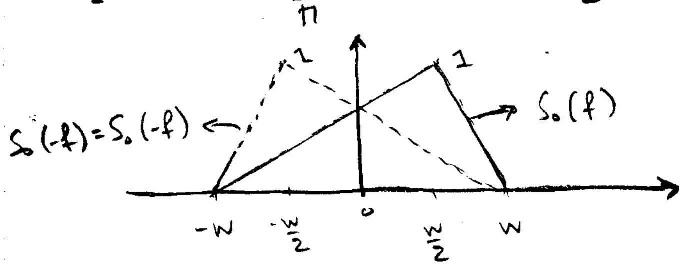
$$= \frac{\sqrt{2}}{\pi}\, j\left(\frac{S_0(f) - S_0^*(-f)}{2}\right) = \frac{\sqrt{2}}{\pi}\, j\, \text{Odd}\left(S_0(f)\right)$$

So $S_I(f)$, $S_Q(f)$ fit definition of quadrature demodulation.

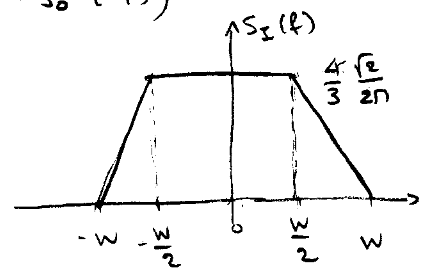
b). $$S_0(f) = \begin{cases} \frac{2}{3W}(f+W) & -W < f \le W/2 \\ -\frac{2}{W}(f-W) & \frac{W}{2} < f \le W \end{cases}$$



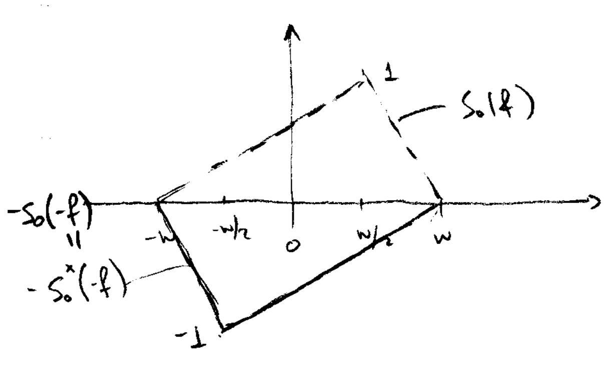
$$S_I(f) = \frac{\sqrt{2}}{\pi}\,\text{Even}\left[S_0(f)\right] = \frac{\sqrt{2}}{2\pi}\left(S_0(f) + S_0^*(-f)\right)$$



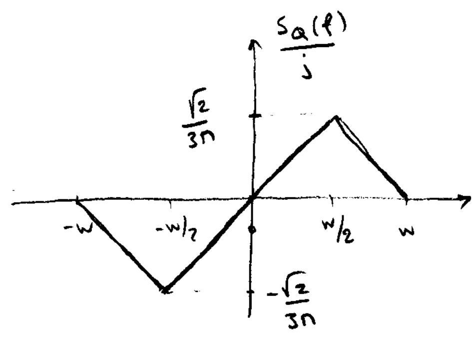
$\Rightarrow$



$$S_Q(f) = \frac{\sqrt{2}}{\pi}\, j\,\text{Odd}\left[S_0(f)\right] = j\,\frac{\sqrt{2}}{2\pi}\left(S_0(f) - S_0^*(-f)\right)$$



$\Rightarrow$



Problem 3

a) In general if $g(x, y)$ and $h(x, y)$ are continuous and differentiable functions then the joint pdf of $z = g(x, y)$ and $w = h(x, y)$ is derived by following three steps:

a) solve system: $\left.\begin{array}{l} g(x, y) = z \\ h(x, y) = w \end{array}\right\} \Rightarrow \begin{array}{l} x_i = \bar{g}_i(z, w) \\ y_i = \bar{h}_i(z, w) \end{array}$ $i = 1 \ldots n$

b) find Jacobian $J_i(z, w) = \begin{vmatrix} \frac{\partial \bar{g}_i}{\partial z} & \frac{\partial \bar{g}_i}{\partial w} \\ \frac{\partial \bar{h}_i}{\partial z} & \frac{\partial \bar{h}_i}{\partial w} \end{vmatrix}$ for each i

c) $f_{zw}(z, w) = \sum_{i=1}^{n} f_{xy}(\bar{g}_i(z,w), \bar{h}_i(z,w)) \cdot |J_i(z,w)|$

abs. value

In our case: $r = \sqrt{x_1^2 + x_2^2} = g(x_1, x_2)$
$\vartheta = \tan^{-1}\left(\frac{x_2}{x_1}\right) = h(x_1, x_2)$

Obviously the system $\left\{\begin{array}{l} r = \sqrt{x_1^2 + x_2^2} \\ \vartheta = \tan^{-1}\left(\frac{x_2}{x_1}\right) \end{array}\right.$ has only one solution: $x_1 = r\cos\vartheta = \bar{g}(r, \vartheta)$
$x_2 = r\sin\vartheta = \bar{h}(r, \vartheta)$

when $\vartheta \in (-\pi, \pi)$.

The Jacobian is $J(z, w) = \begin{vmatrix} \frac{\partial r\cos\vartheta}{\partial r} & \frac{\partial r\cos\vartheta}{\partial \vartheta} \\ \frac{\partial r\sin\vartheta}{\partial r} & \frac{\partial r\sin\vartheta}{\partial \vartheta} \end{vmatrix} =$

$= \begin{vmatrix} \cos\vartheta & -r\sin\vartheta \\ \sin\vartheta & r\cos\vartheta \end{vmatrix} = r\cos^2\vartheta + r\sin^2\vartheta = r$

Thus, $\boxed{f_{r\vartheta} = r \cdot f_{xy}(r\cos\vartheta, r\sin\vartheta)}$

we know that x_1, x_2 are iid $N(0,\sigma^2)$ so

$$f_{X_1 X_2}(x_1,x_2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x_1^2}{2\sigma^2}} \cdot \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x_2^2}{2\sigma^2}} =$$

$$= \frac{1}{2\pi\sigma^2} e^{-\frac{x_1^2+x_2^2}{2\sigma^2}} \qquad x_1, x_2 \in R$$

thus, $$f_{r\vartheta}(r,\vartheta) = \begin{cases} \frac{r}{2\pi\sigma^2} e^{-\frac{r^2}{2\sigma^2}}, & r \geq 0, \ \vartheta \in (-\pi,\pi) \\ 0, & o.w. \end{cases}$$

$$f_r(r) = \int_{-\pi}^{\pi} \frac{r e^{-r^2/2\sigma^2}}{2\pi\cdot\sigma^2} d\vartheta = \frac{r e^{-r^2/2\sigma^2}}{\sigma^2}, \quad r \geq 0 \quad \text{(Rayleigh dist)}$$

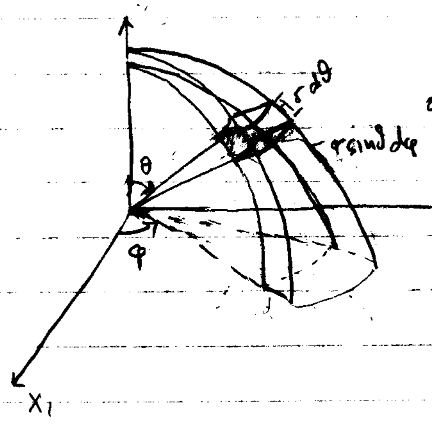
$$f_\vartheta(\vartheta) = \int_0^\infty \frac{r}{2\pi\sigma^2} e^{-r^2/2\sigma^2} dr = \frac{-1}{2\pi} e^{-r^2/2\sigma^2}\Big|_0^\infty = \frac{1}{2\pi}, \quad \vartheta \in (-\pi,\pi) \quad \text{(Uniform dist.)}$$

Note that $f_{r,\theta}(r,\vartheta) = \frac{1}{2\pi} \cdot \frac{r}{\sigma^2} e^{-r^2/2\sigma^2} = f_\vartheta(\vartheta) f_r(r)$

so r, ϑ are independent !!

b) Generalizing to 3-D where

$$\begin{aligned} r &= \sqrt{x_1^2+x_2^2+x_3^2} & x_1 &= r\sin\vartheta\cos\varphi, & r &\geq 0 \\ \vartheta &= \tan^{-1}\left(\sqrt{x_1^2+x_2^2}/x_3\right) \Rightarrow & x_2 &= r\sin\vartheta\sin\varphi, & \vartheta &\in (0,\pi) \\ \varphi &= \tan^{-1}(x_2/x_1) & x_3 &= r\cos\vartheta, & \varphi &\in (0,2\pi) \end{aligned}$$



$$dV = dr\, r d\vartheta\, r\sin\theta\, d\varphi = r^2 \sin\vartheta\, dr\, d\vartheta\, d\varphi$$

$$J(r,\vartheta,\varphi) = \begin{vmatrix} \frac{\partial r\sin\vartheta\cos\varphi}{\partial r} & \frac{\partial r\sin\vartheta\cos\varphi}{\partial\vartheta} & \frac{\partial r\sin\vartheta\cos\varphi}{\partial\varphi} \\ \frac{\partial r\sin\vartheta\sin\varphi}{\partial r} & \frac{\partial r\sin\vartheta\sin\varphi}{\partial\vartheta} & \frac{\partial r\sin\vartheta\sin\varphi}{\partial\varphi} \\ \frac{\partial r\cos\vartheta}{\partial r} & \frac{\partial r\cos\vartheta}{\partial\vartheta} & \frac{\partial r\cos\vartheta}{\partial\varphi} \end{vmatrix} =$$

$$= \begin{vmatrix} \sin\vartheta\cos\varphi & -r\cos\vartheta\cos\varphi & -r\sin\vartheta\sin\varphi \\ \sin\vartheta\sin\varphi & r\cos\vartheta\sin\varphi & r\sin\vartheta\cos\varphi \\ \cos\vartheta & -r\sin\vartheta & 0 \end{vmatrix} = -r^2\sin\vartheta$$

since $f_{x_1x_2x_3}(x_1,x_2,x_3) = \frac{1}{(2\pi\sigma^2)^{3/2}} e^{-\frac{x_1^2+x_2^2+x_3^2}{2\sigma^2}}$

we have:

$$\boxed{f_{r\vartheta\varphi}(r,\vartheta,\varphi) = \frac{r^2\sin\vartheta}{(2\pi\sigma^2)^{3/2}} e^{-r^2/2\sigma^2}} \quad \begin{array}{l} r \geq 0 \\ \vartheta\in(0,\pi) \\ \varphi\in(0,2\pi) \end{array}$$

The marginal distributions are:

$$f_r(r) = \int_0^{\pi}\int_0^{2\pi} \frac{r^2\sin\vartheta}{(2\pi\sigma^2)^{3/2}} e^{-r^2/2\sigma^2}\, d\varphi\, d\vartheta = \sqrt{\frac{2}{\pi}}\frac{r^2}{\sigma^3} e^{-r^2/2\sigma^2}, \; r\geq 0$$

$$f_\vartheta(\vartheta) = \int_0^{2\pi}\int_0^{\infty} \frac{r^2\sin\vartheta}{(2\pi\sigma^2)^{3/2}} e^{-r^2/2\sigma^2}\, dr\, d\varphi =$$

$$= \frac{\sin\vartheta}{\sqrt{2\pi}\,\sigma^3} \underbrace{\int_0^{\infty} r^2 e^{-r^2/2\sigma^2} dr}_{\text{integration by parts}} = \frac{\sin\vartheta}{2} \quad \vartheta\in(0,\pi)$$

$$r^2 e^{-r^2/2\sigma^2} = r\left(-e^{-r^2/2\sigma^2}\sigma^2\right)'$$

$$f_\varphi(\varphi) = \int_0^{\infty}\int_0^{\pi} \frac{r^2\sin\vartheta}{(2\pi\sigma^2)^{3/2}} e^{-r^2/2\sigma^2}\, d\vartheta\, dr = \int_0^{\infty} \frac{r^2 e^{-r^2/2\sigma^2}}{(2\pi\sigma^2)^{3/2}}\, dr =$$

$$= \frac{1}{2\pi} \quad \varphi\in(0,2\pi)$$

Note that $f_{r\theta\varphi}(r,\theta,\varphi) = f_r(r)\, f_\theta(\theta)\, f_\varphi(\varphi)$
so r, θ, φ are statistically independent.

A chi-square random variable is related to a Gaussian r.v. by the following transformation

$$Y = \sum_{i=1}^{n} X_i^2 \quad \text{where} \quad X_i \ \text{iid} \sim N(0,\sigma^2)$$

We say that Y is central chi-square distributed with n degrees of freedom.

We know the pdf of $\sqrt{Y} = r = \sqrt{X_1^2 + X_2^2 + X_3^2}$ and we want to relate it with the pdf of Y ($n = 3$).

We have $Y = r^2 = g(r)$ which has unique solution $r = \sqrt{Y}$.
Using the general formula

$$f_Y(y) = \sum_{i=1}^{n} \frac{f_r(r_i)}{|g'(r_i)|} \quad \text{we have:}$$

$$f_Y(y) = \left.\frac{f_r(r)}{2r}\right|_{r=\sqrt{y}} = \frac{1}{2}\sqrt{\frac{2}{\pi}}\,\frac{y\, e^{-y/2\sigma^2}}{\sigma^3\sqrt{y}} = \underbrace{\frac{y^{1/2}\, e^{-y/2\sigma^2}}{\sqrt{\frac{\Gamma}{2}}\,\sigma^3}}_{\chi^2 \text{ pdf with } 3 \text{ d.o.f.}}$$

so $\boxed{f_r(r) = 2r\, f_Y(r^2)}$

where $Y \sim \chi^2$ with 3 d.o.f.

Interpretation of $f_\theta(\theta)$: In cartesian coordinates if you fix the distance $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$ then $f_{X_1X_2X_3}$ is constant no matter the altitude. In polar coordinates if you fix r then $f_{r\theta\varphi}$ depends on the altitude (varies as $\sin\theta$).

c) If we consider $x_1 \sim N(m\cos\varphi_m, \sigma^2)$

and $x_2 \sim N(m\sin\varphi_m, \sigma^2)$ then the joint density of x_1, x_2 is:

$$f_{x_1x_2}(x_1,x_2) = \frac{1}{2\pi\sigma^2} e^{-\frac{(x_1 - m\cos\varphi_m)^2}{2\sigma^2}} e^{-\frac{(x_2 - m\sin\varphi_m)^2}{2\sigma^2}}$$

$$= \frac{1}{2\pi\sigma^2} e^{-\frac{(x_1^2 + x_2^2 - 2x_1\cos\varphi_m - 2x_2\sin\varphi_m + m^2)}{2\sigma^2}}$$

we have $x_1 = r\cos\vartheta$, $x_2 = r\sin\vartheta$ and we are seeking the joint pdf of r, ϑ.
Following the same procedure as in part (a) we get:

$$f_{r\theta}(r,\vartheta) = r \cdot f_{x_1x_2}(r\cos\vartheta, r\sin\vartheta) =$$

$$= \frac{r}{2\pi\sigma^2} e^{-\left(r^2+m^2-2rm\cos(\vartheta-\varphi_m)\right)/2\sigma^2} \qquad r \geq 0, \quad \vartheta \in (-\pi,\pi)$$

The marginal densities are:

$$f_r(r) = \int_{-\pi}^{\pi} \frac{r}{2\pi\sigma^2} e^{-\left(r^2+m^2-2rm\cos(\vartheta-\varphi_m)\right)/2\sigma^2} d\vartheta$$

$$= \frac{r}{2\pi\sigma^2} e^{-\frac{r^2+m^2}{2\sigma^2}} \int_{-\pi}^{\pi} e^{rm\cos(\vartheta-\varphi_m)/\sigma^2} d\vartheta$$

$$= \frac{r}{2\pi\sigma^2} e^{-\frac{r^2+m^2}{2\sigma^2}} \int_0^{2\pi} e^{rm\cos(\vartheta-\varphi_m+\pi)/\sigma^2} d\vartheta =$$

$$= \frac{r}{2\pi\sigma^2} e^{-\frac{r^2+m^2}{2\sigma^2}} \int_0^{2\pi} e^{rm\cos(\vartheta-\varphi_m)/\sigma^2} d\vartheta$$

$$= \frac{r}{\sigma^2} \cdot e^{\frac{-(r^2+m^2)}{2\sigma^2}} I_0\left(\frac{rm}{\sigma^2}\right), \quad r \geq 0$$

(note that $f_r(r)$ doesn't depend on φ_m).

where we identify the integral

$$I_0(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{x\cos(\vartheta-\varphi)}\, d\vartheta$$

as the modified Bessel function of the 1^{st} kind and zeroth order.

$$f_\vartheta(\vartheta) = \int_0^\infty \frac{r}{2\pi\sigma^2}\, e^{-(r^2+m^2-2rm\cos(\vartheta-\varphi_m))/2\sigma^2}\, dr$$

complete the square to obtain

$$f_\vartheta(\vartheta) = \int_0^\infty \frac{r}{2\pi\sigma^2}\, e^{-\frac{1}{2\sigma^2}(r-m\cos(\vartheta-\varphi_m))^2}\, e^{-\frac{m^2}{2\sigma^2}(1-\cos^2(\vartheta-\varphi_m))}\, dr$$

$$= \frac{e^{-m^2\sin^2(\vartheta-\varphi_m)/2\sigma^2}}{2\pi\sigma^2}\int_0^\infty r\, e^{-(r-m\cos(\vartheta-\varphi_m))^2/2\sigma^2}\, dr$$

doing integration by parts and identifying that

$$\mathrm{erf}(x) = \int_0^x e^{-t^2}\, dt \qquad \text{(called the "error function")}$$

we have the final result:

$$f_\theta(\theta) = \frac{e^{-m^2\sin^2(\vartheta-\varphi_m)/2\sigma^2}}{2\pi\sigma^2}\left[\sigma^2 e^{-m^2\cos^2(\vartheta-\varphi_m)/2\sigma^2} + \sqrt{\frac{\pi}{2}}\,\sigma m\cos(\vartheta-\varphi_m)\cdot\left(1-\mathrm{erf}\left(\frac{-m\cos(\vartheta-\varphi_m)}{\sqrt{2}\,\sigma}\right)\right)\right]$$

d) if we assume φ_m is random then in part (c) all the dencities we found are conditioned with respect to φ_m. Hence, we know $f_{r|\varphi_m}(r)$ and we are looking for $f_r(r)$.
Invoking total probability thm we have.

$$f_r(r) = \int_0^{2\pi} f_{r/\varphi_m}(r)\, f_{\varphi_m}(\varphi)\, d\varphi =$$

$$= \int_0^{2\pi} \frac{r}{\sigma^2} e^{-\left(\frac{r^2+m^2}{2\sigma^2}\right)} I_0\left(\frac{rm}{\sigma^2}\right) \frac{1}{2\pi}\, d\varphi =$$

$$= \frac{r}{\sigma^2} e^{-\left(\frac{r^2+m^2}{2\sigma^2}\right)} I_0\left(\frac{rm}{\sigma^2}\right)$$

Since $f_{r/\varphi_m}(r) = f_r(r)$

then r is independent of φ_m !

Problem 4

a) $$H(f) = \frac{A}{R} e^{-j\,\mathrm{sgn}(f)\,(2\pi f)^2 \frac{R}{g}}$$

note that A is complex $\Rightarrow$ $A = |A| e^{j\varphi}$

thus the phase of the transfer function is:

$$\measuredangle H(f) = \varphi - \mathrm{sgn}(f)\,(2\pi f)^2 \frac{R}{g}$$

<u>phase delay</u>: $T_p(f) \triangleq -\frac{1}{2\pi f} \measuredangle H(f) = \frac{-\varphi + \mathrm{sgn}(f)(2\pi f)^2 R/g}{2\pi f}$

<u>group delay</u>. $T_g(f) \triangleq -\frac{1}{2\pi} \frac{d \measuredangle H(f)}{df} =$

$$= \frac{1}{2\pi}\left(\mathrm{sgn}(f)'\,(2\pi f)^2 \frac{R}{g} + \mathrm{sgn}(f)\; 4\pi f \cdot 2\pi \cdot \frac{R}{g}\right)$$

$\mathrm{sgn}'(f) = 2\delta(f)$

$$= \frac{1}{2\pi} \cdot 2\underbrace{\delta(f)\,(2\pi f)^2}_{0} \frac{R}{g} + \mathrm{sgn}(f)\; 2\pi f\, \frac{R}{g}$$

$$= \mathrm{sgn}(f)\; 4\pi f\, \frac{R}{g}.$$

b) from lecture in order for dispersion to be small $\pi \beta(f) W^2 \leq \frac{\pi}{4}$ where $\beta(f)$ = dispersion.

$$\beta(f) \triangleq -\frac{1}{2\pi} \frac{d^2 \measuredangle H(f)}{df} = \frac{1}{2\pi}\left(\mathrm{sgn}(f)\; 8\pi^2 f\, \frac{R}{g}\right)' =$$

$$= \frac{1}{2\pi}\left(2\underbrace{\delta(f)\; 4\pi f}_{0}\, \frac{R}{g} + \mathrm{sgn}(f)\; 8\pi^2 \frac{R}{g}\right) = 4\pi\, \mathrm{sgn}(f)\, \frac{R}{g}$$

So the condition is $2\,\mathrm{sgn}(f)\, \frac{R}{g} W^2 \leq \frac{1}{4} \xrightarrow{f \geq 0}$

$$W^2 \leq \frac{1}{16\pi} \frac{g}{R} \Rightarrow W \leq \left(\frac{g}{16\pi R}\right)^{1/2}$$

so bandwidth is inversely proportional to $R^{1/2}$.

c) $x(t) = Ce^{-\kappa t}\left(t\cos(2\pi f_0 t) - \sin(2\pi f_0 t)\right) u(t)$

$= Ce^{-\kappa t}\left(t\cdot\cos(2\pi f_0 t) - 1\cdot\sin(2\pi f_0 t)\right) u(t)$

$= Ce^{-\kappa t} \,\mathrm{Re}\left[(t+j)\left(\cos(2\pi f_0 t) + j\sin(2\pi f_0 t)\right)\right] u(t) =$

$= 2\,\mathrm{Re}\left[\frac{C}{2} e^{-\kappa t}(t+j)\,u(t)\, e^{j2\pi f_0 t}\right] \Rightarrow \tilde{x}_0(t) = \frac{C}{2}e^{-\kappa t}(t+j)\,u(t)$

following the convention $\int |\tilde{x}(t)|^2 dt = 1 \Rightarrow$

$\int_{-\infty}^{\infty} \frac{C^2}{4} e^{-2\kappa t}(t^2+1)\,u^2(t)\,dt = 1 \Rightarrow \frac{C^2}{4}\int_0^{\infty} e^{-2\kappa t}(t^2+1)\,dt = 1$

$\Rightarrow C = \sqrt{\frac{16\kappa^3}{1+2\kappa^2}}$

$X_0(f) = \mathcal{F}\left\{\frac{C}{2}e^{-\kappa t}(t+j)\,u(t)\right\} =$

$\frac{C}{2}\left[\mathcal{F}\left\{e^{-\kappa t}\,t\,u(t)\right\} + j\,\mathcal{F}\left\{e^{-\kappa t}u(t)\right\}\right]$

$= \frac{C/2}{(\kappa+j2\pi f)^2} + \frac{jC/2}{\kappa+j2\pi f}$

d) neglecting dispersion, the complex envelope of the output is:

$\tilde{y}_0(t) = |H(f_0)|\, e^{-j2\pi f_0 T_p(f_0)}\, \tilde{x}_0(t - T_g(f_0))$

$= \frac{|A|}{R} e^{j\left(\varphi - (2\pi f_0)^2 \frac{R}{g}\right)} \frac{C}{2} e^{-\kappa\left(t - 4\pi f_0 \frac{R}{g}\right)}\left(t - 4\pi f_0\frac{R}{g} + j\right) u\left(t - 4\pi f_0 \frac{R}{g}\right)$

thus the narrowband output is

$y(t) = 2\,\mathrm{Re}\left[\tilde{y}_0(t)\, e^{j2\pi f_0 t}\right]$

when $\quad 2\pi f_0 \frac{R}{g} = \frac{1001}{2f_0} \Rightarrow f_0^2 = \frac{g\cdot 1001}{4\pi R}$

$T_g(f_0) = 4\pi f_0 \frac{R}{g} = \frac{1001}{f_0}$. as $f_0 \to \infty \Rightarrow T_g \to 0$

$f_0 \to 0 \Rightarrow T_g \to \infty$

so for high frequencies group delay may be neglected but not for low frequencies.

$$T_p(f_0) = \frac{-\varphi}{2\pi f_0} + 2\pi f_0 \frac{R}{g} = \frac{1}{f_0}\left(\frac{1}{2} - \frac{\varphi}{2\pi}\right)$$

so $\quad 2\pi f_0 T_p(f_0) = 2\pi\left(\frac{1}{2} - \frac{\varphi}{2\pi}\right) = \pi - \varphi$

and $\quad e^{-j2\pi f_0 T_p(f_0)} = e^{j(\varphi-\pi)} = -e^{j\varphi}$

e). Note as t increases $x(t) \sim e^{-Kt}$ decreases exponentially with time. Assuming that the effective duration T of $x(t)$ is when $e^{-Kt} > e^{-1}$ so $T \simeq \frac{1}{K}$ then the bandwidth is approximately $W \simeq K$.

From part (b) we obtained a constraint for the bandwidth in order to neglect dispersion thus:

$$W^2 \le \frac{g}{16\pi R} \Rightarrow \boxed{K^2 \le \frac{g}{16\pi R}}$$