2.163J/6.455J
Sonar, Radar and Seismic Signal Processing

Problem Set 3

Issued: October 23, 2006
Due: November 6, 2006

Please note that Quiz I will be distributed on November 6, 2006 and will be due on November 20, 2004. Also, this problem set involves more than routine Fourier transforms operations relating correlation functions and power spectral densities, so each problem may require a bit more time.

Problem 1: Detection of Gaussian signals in Gaussian noise.

The deflection metric d^2 is often used for specifying the performance when detecting a gaussian signal imbedded in gaussian noise. For this, we have two hypotheses, H_0 and H_1. On the noise only hypothesis we have

$$\mathbf{r} = \mathbf{w}$$

where $\mathbf{w}$ is an N dimensional, complex gaussian random vector whose components are each complex, gaussian with mean 0 and variance σ_w^2. The components are statistically independent. On the signal present hypothesis, H_1, the observation vector is given by

$$\mathbf{r} = \mathbf{s} + \mathbf{w}$$

where $\mathbf{s}$ is a N dimensional complex, gaussian, random vector whose components are each complex, gaussian random variables with mean 0 and variance $\lambda_n,\ n = 1, N$.

The deflection statistic is given by

$$L = |r|^2$$

and the deflection metric is given by

$$d^2 = \frac{[(E(L|H_1) - E(L|H_0))^2]}{\sigma_L^2}$$

1) Find the detection metric d^2

2) Consider that the total power is the signal is constrainted such that

$$\sum_{n=1}^{N} \lambda_n = E_s$$

Find the optimal choice of N.

Please note that these are complex, gaussian vectors! Also, note that this metric d^2 is an usefuk approximation for the perforance of a detection system when $\lambda_n/\sigma_w^2 < 1$. A little more work with this allows us to determine the performance of a radar/sonar system.

Problem 2: 2nd moments with a non Gaussian process

In this problem we consider a process which "switches" between two types of waveforms. We construct this with a model

$$y(t) \;=\; s(t)a(t) \;+\; (1 - s(t))b(t)$$

where

$s(t)$ is the Telegrapher's wave which switches between $[0, 1]$ at Poisson intervals with rate λ;

$a(t)$ and $b(t)$ are wide sense stationary, zero mean, random process with correlation functions

$$R_a(\tau) \;=\; P_a \cos(2\pi f_a \tau) e^{-(\frac{|\tau|}{T})}$$

$$R_b(\tau) \;=\; P_a \cos(2\pi f_b \tau) e^{-(\frac{|\tau|}{T})}$$

Recall that the Telegrapher's wave has an autocorrelation given by

$$R_s(\tau) \;=\; \frac{1}{4}(e^{-\lambda|\tau|} + 1)$$

1) Is the process $y(t)$ wide sense stationary? If so, determine the spectral density function, $S_x(\omega)$.

2) For illustration purposes assume $f_b \approx 10 f_a,\ 1 >> T\lambda,\ f_a,\ f_b >> 1/T$ and $\lambda << f_a$ Sketch the spectral density function with parameters well labeled and an illustrative sample function of the ensemble. This is to test your understanding of the physical consequences of the model. [1]

[1] This is a more realistic model of the Telegrapher's signal where the audible signal switched between two different tonals. Note all the transforms can be done analytically, but a sketch of the PSD might help you.

Problem 3: Difference equations (You will need Matlab for this problem)

We have used the notation

$$\sum_{k=0}^{N} a_k y[n-k] \;=\; \sum_{k=0}^{M} b_k x[n-k]$$

for representing difference equations. In this problem M=N=4. For convenience we represent the coefficients in column vector form where

$$b \;=\; [\;\; 0.0675 \quad 0.0000 \quad -.1349 \quad 0.0000 \quad 0.0675 \;\;]$$

$$a \;=\; [\;\; 1.0000 \quad -1.9425 \quad 2.1192 \quad -1.2167 \quad 0.4128 \;\;]$$

i) Plot $H(e^{j\omega})$. Note this includes the both magnitude and phase.

ii) The phase and group delays through a digital processor is defined in the same manner as for a continuous time filter. Plot both delays *versus* frequency.

iii) Implement the difference equation when the input, $x[n]$ is the unit impulse sequence. (You stop the recursion when the response is less than one percent of its maximum response.)

iv) Use the polynomial root finder in MATLAB to determine the poles and zeros of the difference equation.

Problem 4: Transforms to the discrete domain: (This is an example of how to reconcile continuous and discrete time processes.)

Consider a variant of our well known friend the damped harmonic oscillator with

$$H(s) \; = \; \frac{s\omega_o}{s^2 + 2\delta\omega_o s + \omega_o^2}$$

with the resonant period given by $T_o \; = \; \frac{2\pi}{\omega_o}$

i) Determine the corresponding difference equation when the derivative operations are given numerically as

$$\frac{dx(t)}{dt} \quad \longrightarrow \quad \frac{x[n+1]-x[n-1]}{2\Delta T}$$

$$\frac{d^2x}{dt^2} \quad \longrightarrow \quad \frac{x[n+1]-2x[n]+x[n-1]}{2\Delta T^2}$$

ii) Plot the solution to the difference equation and compare it to the impulse response of the continuous time system for $\Delta T = .001, \; .01, \; .1$ x T_o.

iii) Consider the samples data and bilinear mappings to the Z plane for the same values of ΔT. Find the pole/zero loci and the DTFT. Which mapping come closer to representing the fundamental properties of the continuous system such as peak response, decay rate, and group delays.

Much of this problem can be down analytically, but the exact pole/zero locations, the transfer function and the group delay eventually need to done numerically.

Problem 5: Blackman-Tukey (indirect) Spectral Estimation

We want to examine the properties of Blackman-Tukey spectral estimation using the Hermite weighting given by: [2]

$$w(\tau) \;=\; e^{-\frac{1}{2}(\frac{\tau}{M})^2}\left[1 \;+\; (\frac{\tau}{T_H})^2\right]$$

i) Determine the window, $W(f)$, the transform of $w(\tau)$. You may assume that $M << T$, so that end effects in the integration may be ignored.

ii) Consider the problem of estimating the power spectral density when the process is a bandlimited white process given by:

$$S_x(f) \;=\; \left[\begin{array}{rl} \frac{P}{2W} & |f| < W \\ 0 & |f| \geq W \end{array}\right.$$

Discuss the resolution of this spectrum in terms of the normalized quantities $M \cdot W$ and $T_H \cdot W$.

iii) Calculate the window coefficent, C_w, such that the variance in estimating a smooth power density spectrum may be espressed as

$$\frac{\sigma^2_{\hat{S}(f|T)}}{S^2(f)} = C_w \frac{M}{T}$$

Plot this variance as a function of T_H. (You will find the normalized quantity $\frac{T_H}{M}$ a more convenient quantity.)

[2] The transforms required below are tabulated in several mathematical handbooks.