Spectral Estimation

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- In our analysis of random waveforms (stochastic processes) we emphasize differences in temporal domain / frequency domain descriptions of deterministic waveforms versus random waveforms. In particular a deterministic framework focuses on the actual waveform (realization) itself, whereas a stochastic framework focuses on averages which involve realizations of these waveforms:

Deterministic Signals	Random/Stochastic Signals
$x(t)$	$R_x(\tau) = E\{x(t+\tau)x^*(t)\}$
Energy Spectrum: $\lvert X(f)\rvert^2$	Power Density Spectrum: $S_x(f)$
Energy $\int_{-\infty}^{\infty} \lvert x(t)\rvert^2 dt = \int_{-\infty}^{\infty} \lvert X(f)\rvert^2 df$	Average Power $E\{\lvert x(t)\rvert^2\} = R_x(0) = \int_{-\infty}^{\infty} S_x(f) df$
Periodic $x(t+T) = x(t)$	Periodic $R_x(\tau+T) = R_x(\tau)$
LTI Systems $y(t) = x(t) * h(t)$	LTI Systems $R_y(\tau) = R_x(\tau) * h^*(-\tau) * h(\tau)$
$\lvert Y(f)\rvert^2 = \lvert X(f)\rvert^2 \lvert H(f)\rvert^2$	$S_y(f) = S_x(f)\lvert H(f)\rvert^2$

(1)

The autocorrelation function $R_x(\tau)$ plays the role of the actual waveform $x(t)$ and the power spectral density (PSD) function $S_x(f)$ plays the role of the energy spectra $|X(f)|^2$.

- With a focus on averages of waveforms versus waveforms themselves, we note that to obtain "good" averages requires sufficient amounts of data sharing identical statistical properties. In practice the amount of available data is typically quite limited, and often only stationary over finite intervals of time.

- PSD Estimation Problem: From a finite data record (of duration T) of at least WSS data, estimate the power distribution over frequency.

- It is of necessity in PSD estimation that an implicit assumption of first and second moment ergodicity be made. In particular we assume that in the limit of large data record length the following limits hold:

$$\eta_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)dt \tag{2}$$

$$R_x(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)x^*(t-\tau)dt, \tag{3}$$

and defining

$$X_T(f) \triangleq \int_{-T/2}^{T/2} x(t)e^{-j2\pi ft}dt \tag{4}$$

we assume that

$$S_x(f) = \lim_{T\to\infty} E\left\{\frac{1}{T}|X_T(f)|^2\right\}. \tag{5}$$

- The quantity $\frac{1}{T}|X_T(f)|^2$ is known as the *periodogram*. More said later.
- Before we discuss two classical techniques of PSD estimation, we first review some important properties of estimators in general. The goal is to specify what properties are desirable properties for estimators.

Properties of Estimators

- say $\widehat{a}$ is an estimate of a deterministic but unknown parameter a
  - In general the estimate $\widehat{a}$ is determine from some observed random data, that is $\widehat{a} = f(\text{RANDOM DATA})$, and therefore $\widehat{a}$ is itself a random variable
- The Mean squared error (MSE) is a common, and useful measure of performance. Note from the following that it consists of contributions from two major sources:

$$\begin{aligned}
\text{MSE} &= E\{|\widehat{a} - a|^2\} \\
&= E\{|\widehat{a} - E\{\widehat{a}\} + E\{\widehat{a}\} - a|^2\} \\
&= E\{|\widehat{a} - E\{\widehat{a}\}|^2\} + |E\{\widehat{a}\} - a|^2 + 2\text{Re}\, E\left\{[(\widehat{a} - E\{\widehat{a}\})\,(E\{\widehat{a}\} - a)^*]\right\} \\
&= E\{|\widehat{a} - E\{\widehat{a}\}|^2\} + |E\{\widehat{a}\} - a|^2 \\
&= \sigma^2_{\widehat{a}} + |\text{Bias}\{\widehat{a}\}|^2.
\end{aligned} \tag{6}$$

- Both the variance and bias of an estimator contribute to the overall MSE. Consequently, analysis of estimators typically focuses on these two quantities as indicative measures of performance. We illustrate some common properties of estimators involving the bias and variance via a simple coin toss experiment.

Heads or Tails?

- Consider the well-known coin flip experiment. Assume that the probability of heads is given by p, and therefore that of tails is given by $1 - p$. Consider the goal of estimating the parameter p, the probability of heads.
- Clearly this experiment is well modeled by a Bernoulli random variable x, with pdf

$$f_x(x_0) = p \cdot \delta(x_0 - 1) + (1 - p) \cdot \delta(x_0). \tag{7}$$

It has the following moments:

$$\begin{aligned} E\{x\} &= 0 \cdot (1-p) + 1 \cdot p = p \\ E\{x^2\} &= 0^2 \cdot (1-p) + 1^2 \cdot p = p \\ \sigma_x^2 &= p(1-p), \end{aligned} \tag{8}$$

- Given n independent observations of this Bernoulli experiment $x_1, \ldots, x_n$, consider the following estimator of parameter p (sample mean):

$$\widehat{p}_n \triangleq \frac{1}{n} \sum_{i=1}^{n} x_i \tag{9}$$

and note that it has the following mean and variance:

$$\begin{aligned} E\{\widehat{p}_n\} &= p \\ \sigma_{\widehat{p}_n}^2 &= \frac{p(1-p)}{n} = \frac{\sigma_x^2}{n}. \end{aligned} \tag{10}$$

- We say that $\widehat{p}_n$ is an *unbiased* estimator of parameter p because $E\{\widehat{p}_n\} = p$. If $E\{\widehat{p}_n\} \neq p$, then we would say that $\widehat{p}_n$ is a *biased* estimator of parameter p.

- If $E\{\widehat{p}_n\} \neq p$, but $\lim_{n\to\infty} E\{\widehat{p}_n\} = p$, then we would say that $\widehat{p}_n$ is *asymptotically unbiased.*

- Note the normalized variance $\sigma^2 \triangleq \sigma_{\widehat{p}_n}^2 / \sigma_x^2 = 1/n$. It is a function of parameters we can control/design.

- Confidence Intervals:

  - When n is large we can approximate the pdf of $\widehat{p}_n$ via the central limit theorem as

$$\widehat{p}_n \sim N\left(p, \frac{p(1-p)}{n}\right), \tag{11}$$

  from which one can compute an α level confidence interval given by

$$\Pr\left(\frac{|\widehat{p}_n - p|}{\sigma_{\widehat{p}_n}} \leq T_\alpha\right) = \alpha. \tag{12}$$

  - Given an estimate $\widehat{p}_n$, we can say with $100 \cdot \alpha\%$ certainty that the true value of the parameter p will fall in the interval

$$[\widehat{p}_n - T_\alpha, \widehat{p}_n + T_\alpha]. \tag{13}$$

  Hence, the name confidence intervals.

- Note from the above approximate pdf for $\widehat{p}_n$, or simply from the fact that $\widehat{p}_n$ is unbiased with a variance that approaches zero as n increases, that $\lim_{n\to\infty} \Pr\left(|\widehat{p}_n - p| > \epsilon\right) = 0$. When this limit holds for an estimator we say that $\widehat{p}_n$ is a *consistent* estimator of parameter p.

- Given two different unbiased estimators $\widehat{p1} = f_1(x_1, \ldots, x_n)$ and $\widehat{p2} = f_2(x_1, \ldots, x_n)$ of the same parameter p, we say that $\widehat{p1}$ is more *efficient* than $\widehat{p2}$ if $\sigma^2_{\widehat{p1}} < \sigma^2_{\widehat{p2}}$.

- The most efficient unbiased estimator among all possible estimators is the one which obtains the *Cramér-Rao Lower Bound* (CRB) on unbiased estimators. It can be shown that for any unbiased estimator $\widehat{p}$ of parameter p that

$$\sigma^2_{CRB}(p) \leq \sigma^2_{\widehat{p}}. \tag{14}$$

- An unbiased estimator $\widehat{p}_n$ which obtains the CRB, *i.e.* $\sigma^2_{\widehat{p}_n} = \sigma^2_{CRB}(p)$ is said to be an *efficient* estimator.

- If $\sigma^2_{\widehat{p}_n} > \sigma^2_{CRB}(p)$ for finite n, but $\lim_{n\to\infty} \sigma^2_{\widehat{p}_n} = \sigma^2_{CRB}(p)$, then $\widehat{p}_n$ is said to be *asymptotically efficient*.

Blackman-Tukey Method / Indirect Method

- The Blackman-Tukey (BT) Method of Power Spectral Density Estimation can be summarized in three steps:

  1. Estimate $R_x(\tau) \Longrightarrow \widehat{R}_x(\tau : T)$.
  2. Apply window $w_0(\tau)$.
  3. Fourier Transform:

$$\widehat{S}_x(f : T) \stackrel{\triangle}{=} \int_{-\infty}^{\infty} w_0(\tau)\widehat{R}_x(\tau : T)e^{-j2\pi f\tau}d\tau. \tag{15}$$

- Note that the BT method is often called the Indirect method of PSD estimation because one first computes the autocorrelation function estimate $\widehat{R}_x(\tau)$, then transforms to the frequency domain. This is to be contrasted with methods which directly transforms data to the frequency domain prior to any estimation.

- Assume $x(t)$ is a real zero mean Gaussian process. Let $x_T(t)$ be $x(t)$ observed for a finite duration T:

$$x_T(t) = \begin{cases} x(t), & -T/2 \leq t \leq T/2 \\ 0, & \text{Otherwise.} \end{cases} \tag{16}$$

- Consider the following estimate of the process autocorrelation:

$$\boxed{\widehat{R}_x(\tau : T) = \frac{1}{T}\int_{-T/2}^{T/2} x(t)x(t-\tau)dt = \frac{1}{T}\int_{-\infty}^{\infty} x_T(t)x_T(t-\tau)dt} \tag{17}$$

- Note that

$$\widehat{R}_x(\tau : T) = \frac{1}{T}x_T(\tau) * x_T^*(-\tau) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{T}X_T(f)X_T^*(f) = \frac{1}{T}|X_T(f)|^2 \tag{18}$$

  Thus, BT with no window coincides with classical periodogram.

- Note that $\widehat{R}_x(\tau : T)$ is a function of a stochastic process. Hence, it is itself a random process. It has a mean function $\eta_{\widehat{R}_x}(\tau : T)$ and covariance function $K_{\widehat{R}_x}(\tau_1, \tau_2 : T)$.

- Ultimately, we are interested in the statistics (mean and covariance) of the resulting PSD estimate $\widehat{S}_x(f : T)$. We will find, however, that the mean and covariance of $\widehat{R}_x(\tau : T)$ need be determined first.

  It is straight forward to show (see figure 1) that the mean function is given by

$$\boxed{\eta_{\widehat{R}_x}(\tau : T) = E\left\{\widehat{R}_x(\tau : T)\right\} = \left(1 - \frac{|\tau|}{T}\right) \cdot R_x(\tau).} \tag{19}$$

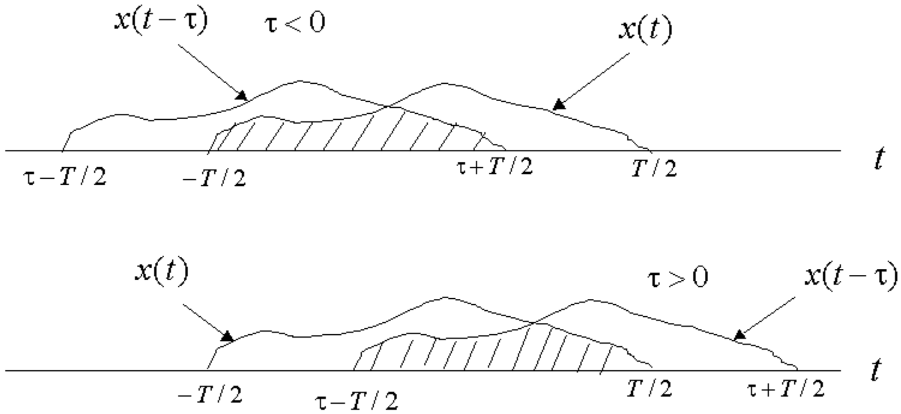

Figure 1: Time shifted waveforms of $\widehat{R}_x(\tau : T)$

Hence, the bias is given by

$$\text{BIAS}_{\widehat{R}_x} = E\left\{\widehat{R}_x(\tau : T)\right\} - R_x(\tau) = -\frac{|\tau|}{T} R_x(\tau). \tag{20}$$

Note that

- $\widehat{R}_x(\tau : T)$ is biased for finite τ, but approximately unbiased for $|\tau| \ll T$.
- Asymptotically unbiased for all τ as $T \to \infty$.

- By definition the covariance function of this autocorrelation function estimate is given by

$$K_{\widehat{R}_x}(\tau_1, \tau_2 : T) = \tag{21}$$

$$E\left\{\left[\widehat{R}_x(\tau_1 : T) - \left(1 - \frac{|\tau_1|}{T}\right) \cdot R_x(\tau_1)\right]\left[\widehat{R}_x(\tau_2 : T) - \left(1 - \frac{|\tau_2|}{T}\right) \cdot R_x(\tau_2)\right]\right\}$$

$$= E\left\{\widehat{R}_x(\tau_1 : T)\widehat{R}_x(\tau_2 : T)\right\} - \left(1 - \frac{|\tau_1|}{T}\right)\left(1 - \frac{|\tau_2|}{T}\right) \cdot R_x(\tau_1) R_x(\tau_2). \tag{22}$$

The product of functions is given by

$$\widehat{R}_x(\tau_1 : T)\widehat{R}_x(\tau_2 : T) = \int_{\tau_1 - T/2}^{T/2} dt_1 \int_{\tau_2 - T/2}^{T/2} dt_2 \cdot \frac{1}{T^2} \cdot x(t_1)x(t_1 - \tau_1)x(t_2)x(t_2 - \tau_2). \tag{23}$$

Making liberal use of the following moment formula for Gaussian random variables,

$$E\{x_1 x_2 x_3 x_4\} = E\{x_1 x_2\}E\{x_3 x_4\} + E\{x_1 x_3\}E\{x_2 x_4\} + E\{x_1 x_4\}E\{x_2 x_3\} \tag{24}$$

we can find the desired expectation, and obtain the following expression for the covariance function of the autocorrelation estimate

$$K_{\widehat{R}_x}(\tau_1, \tau_2 : T) = \int_{\tau_1 - T/2}^{T/2} dt_1 \int_{\tau_2 - T/2}^{T/2} dt_2 \tag{25}$$

$$\times \frac{1}{T^2} \cdot \left[R_x(t_1 - t_2) R_x(t_1 - t_2 - \tau_1 + \tau_2) + R_x(t_1 - t_2 + \tau_2) R_x(t_1 - t_2 - \tau_1) \right].$$

Making the change of variables:

$$\begin{aligned} t' &= t_1 \\ t &= t_1 - t_2 \end{aligned} \tag{26}$$

we obtain the slight simplication

$$\begin{aligned} K_{\widehat{R}_x}(\tau_1, \tau_2 : T) = \frac{1}{T} \int_{-T}^{T} \left(1 - \frac{|t|}{T}\right) R_x(t) R_x[t - (\tau_1 - \tau_2)] dt + \\ \frac{1}{T} \int_{-T+\tau_2}^{T+\tau_2} \left(1 - \frac{|t|}{T}\right) R_x(t) R_x[t - (\tau_1 + \tau_2)] dt. \end{aligned} \tag{27}$$

If we in addition assume that $R_x(\tau) \approx 0$ for $|\tau| > T_c/2$, which is often the case in the absence of tonals, and that $T_c \ll T$, then we can make the useful approximation:

$$\boxed{K_{\widehat{R}_x}(\tau_1, \tau_2 : T) \simeq \frac{1}{T} \int_{-\infty}^{\infty} R_x(t) R_x[t - (\tau_1 - \tau_2)] + R_x(t) R_x[t - (\tau_1 + \tau_2)] \; dt.} \tag{28}$$

- Note that the covariance function for the estimate $\widehat{R}_x(\tau : T)$ depends on the true covariance function $R_x(\tau)$ and time shifted versions shifted by the difference of lags $(\tau_1 - \tau_2)$ and the sum of lags $(\tau_1 + \tau_2)$.
- An expression for the variance of the estimate $\widehat{R}_x(\tau : T)$ is easily obtained by setting $\tau_1 = \tau_2$:

$$\boxed{\sigma^2_{\widehat{R}_x}(\tau : T) = K_{\widehat{R}_x}(\tau, \tau : T) \simeq \frac{1}{T} \int_{-\infty}^{\infty} R_x^2(t) + R_x(t) R_x(t - 2\tau) \; dt} \tag{29}$$

- Since we assume that that $R_x(\tau) \approx 0$ for $|\tau| > T_c/2$, we can make the following approximation for the covariance of $\widehat{R}_x(\tau : T)$ away from the origin (see figure 2):

$$K_{\widehat{R}_x}(\tau_1, \tau_2 : T) \simeq \begin{cases} \frac{1}{T} \int_{-\infty}^{\infty} R_x(t) R_x[t - (\tau_1 - \tau_2)] dt, & |\tau_1 + \tau_2| > T_c \\ \frac{1}{T} \int_{-\infty}^{\infty} R_x(t) \left\{ R_x[t - (\tau_1 - \tau_2)] + R_x[t - (\tau_1 + \tau_2)] \right\} dt, & |\tau_1 + \tau_2| < T_c \end{cases}.$$

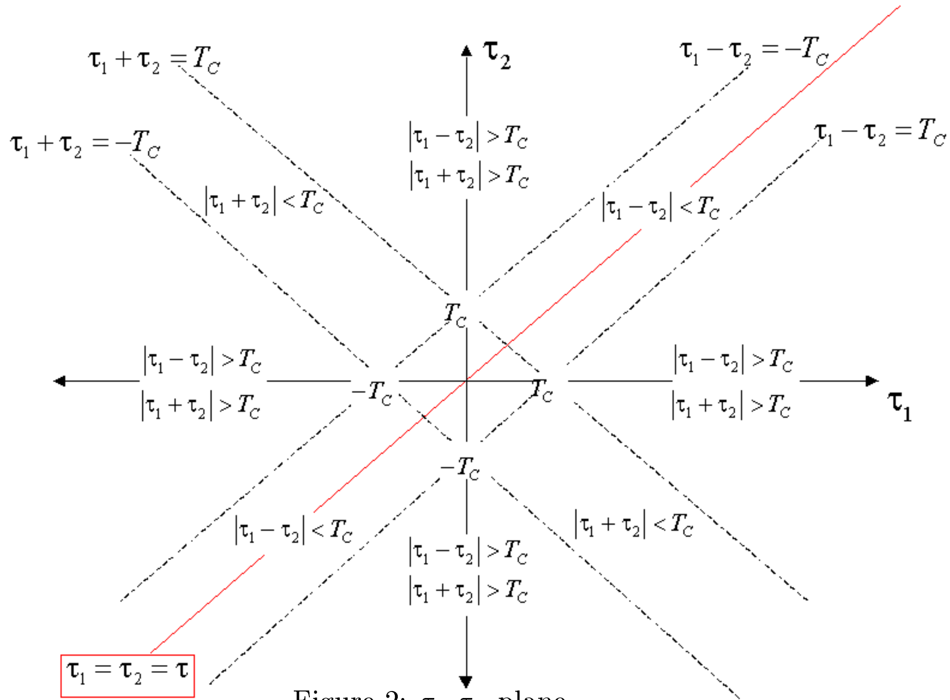

Figure 2: τ_1, τ_2–plane

- Similary, we can approximate its variance away from the origin as well:

$$\sigma^2_{\widehat{R}_x}(\tau : T) \simeq \begin{cases} \dfrac{1}{T}\displaystyle\int_{-\infty}^{\infty} R_x^2(t)dt = \dfrac{1}{T}\int_{-\infty}^{\infty} S_x^2(f)df, & |\tau| > T_c/2 \\ \dfrac{1}{T}\displaystyle\int_{-\infty}^{\infty} R_x^2(t) + R_x(t)R_x(t-2\tau)dt, & |\tau| < T_c/2 \end{cases}. \quad (30)$$

- Recall that the BT estimate of the PSD is obtained by Fourier transforming the estimated autocorrelation function:

$$\boxed{\widehat{S}_x(f : T) = \int_{-\infty}^{\infty} w_0(\tau) \cdot \widehat{R}_x(\tau : T)e^{-j2\pi f\tau}.d\tau} \quad (31)$$

- The mean of this estimate is shown to be

$$\begin{aligned} \eta_{\widehat{S}_x}(f) = E\left\{\widehat{S}_x(f : T)\right\} &= \int_{-\infty}^{\infty} w_0(\tau)\left(1 - \frac{|\tau|}{T}\right)R_x(\tau)e^{-j2\pi f\tau}d\tau \\ &= \int_{-\infty}^{\infty} w(\tau)R_x(\tau)e^{-j2\pi f\tau}d\tau = \int_{-\infty}^{\infty} W(\nu - f)S_x(\nu)d\nu. \end{aligned} \quad (32)$$

Hence, the expected value of the BT PSD estimate is not the true PSD of the process $S_x(f)$, but rather a *smoothed/averaged* version of $S_x(f)$.

- In light of this, it is useful to adopt certain properties of the window function $w(\tau)$:

1. Assume window function of duration M, *i.e.* $w(\tau) = 0$ for $|\tau| > M/2$.
2. Assume window is an even function: $w(\tau) = w(-\tau) \Longrightarrow W(f) = W(-f)$.
3. Assume that $w(0) = 1 \Longrightarrow \int_{-\infty}^{\infty} W(f)df = 1$.

- Concerning the bias of this estimate recall first the sifting property of the Dirac Delta impulse function:

$$S_x(f) = \int_{-\infty}^{\infty} S_x(\nu)\delta(f-\nu)d\nu = \int_{-\infty}^{\infty} S_x(\nu)\delta(\nu-f)d\nu. \tag{33}$$

Thus, the bias is expressed as

$$\boxed{E\left\{\widehat{S}_x(f:T)\right\} - S_x(f) = \int_{-\infty}^{\infty} \left[W(\nu-f) - \delta(\nu-f)\right] S_x(\nu)d\nu.} \tag{34}$$

- The last window function property guarantees that if the true spectrum $S_x(f)$ is constant over the window bandwidth (locally smooth), *i.e.* $S_x(f)$ appears to be a flat/white-noise like process relative to the window bandwidth B_w, then the estimate is unbiased:

$$E\left\{\widehat{S}_x(f:T)\right\} = \int_{-\infty}^{\infty} W(\nu-f)S_x(\nu)d\nu \simeq S_x(f)\int_{-\infty}^{\infty} W(\nu-f)d\nu = S_x(f). \tag{35}$$

- Note from eq(34) that we can reduce the overall bias in the BT PSD estimate by making $W(f)$ look more like an impulse $\delta(f)$. This requires that we increase the window length M.

- Concerning the covariance function of the BT PSD estimate, note that

$$\left[\widehat{S}_x(f_1:T) - E\{\widehat{S}_x(f_1:T)\}\right]\left[\widehat{S}_x(f_2:T) - E\{\widehat{S}_x(f_2:T)\}\right] = \tag{36}$$

$$\int_{-\infty}^{\infty} w(\tau_1)e^{-j2\pi f_1\tau_1}d\tau_1 \int_{-\infty}^{\infty} w(\tau_2)e^{-j2\pi f_2\tau_2}d\tau_2$$

$$\times\left[\widehat{R}_x(\tau_1:T) - \left(1-\frac{|\tau_1|}{T}\right)R_x(\tau_1)\right]\left[\widehat{R}_x(\tau_2:T) - \left(1-\frac{|\tau_2|}{T}\right)R_x(\tau_2)\right].$$

Hence, taking the expectation one obtains

$$K_{\widehat{S}_x}(f_1,f_2:T) = \int_{-\infty}^{\infty} d\tau_1 \int_{-\infty}^{\infty} d\tau_2\ w(\tau_1)w(\tau_2)e^{-j2\pi f_1\tau_1}e^{-j2\pi f_2\tau_2}K_{\widehat{R}_x}(\tau_1,\tau_2:T), \tag{37}$$

or explicitly

$$K_{\widehat{S}_x}(f_1,f_2:T) = \int_{-\infty}^{\infty} d\tau_1 \int_{-\infty}^{\infty} d\tau_2\ w(\tau_1)w(\tau_2)e^{-j2\pi f_1\tau_1}e^{-j2\pi f_2\tau_2}\int_{\tau_1-T/2}^{T/2} dt_1 \int_{\tau_2-T/2}^{T/2} dt_2 \tag{38}$$

$$\times\frac{1}{T^2}\cdot\left[R_x(t_1-t_2)R_x(t_1-t_2-\tau_1+\tau_2) + R_x(t_1-t_2+\tau_2)R_x(t_1-t_2-\tau_1)\right].$$

Recalling the Fourier relationships

$$
\begin{aligned}
R_x(t_1 - t_2) &= \int_{-\infty}^{\infty} S_x(f) e^{j2\pi f(t_1 - t_2)} df \\
R_x(t_1 - t_2 - \tau_1 + \tau_2) &= \int_{-\infty}^{\infty} S_x(f) e^{j2\pi f(t_1 - t_2 - \tau_1 + \tau_2)} df \\
R_x(t_1 - t_2 + \tau_2) &= \int_{-\infty}^{\infty} S_x(f) e^{j2\pi f(t_1 - t_2 + \tau_2)} df \\
R_x(t_1 - t_2 - \tau_1) &= \int_{-\infty}^{\infty} S_x(f) e^{j2\pi f(t_1 - t_2 - \tau_1)} df,
\end{aligned}
\tag{39}
$$

we can write the covariance function as

$$
\begin{aligned}
K_{\widehat{S}_x}(f_1, f_2 : T) = \int_{-\infty}^{\infty} d\tau_1 \int_{-\infty}^{\infty} d\tau_2 \; w(\tau_1) w(\tau_2) e^{-j2\pi f_1 \tau_1} e^{-j2\pi f_2 \tau_2} \int_{-T/2}^{T/2} dt_1 \int_{-T/2}^{T/2} dt_2 \\
\times \frac{1}{T^2} \int_{-\infty}^{\infty} d\nu_1 \int_{-\infty}^{\infty} d\nu_2 S_x(\nu_1) S_x(\nu_2) \\
\times \left[e^{j2\pi\nu_1(t_1 - t_2) - j2\pi\nu_2[(t_1 - t_2) - (\tau_1 - \tau_2)]} + e^{j2\pi\nu_1[(t_1 - t_2) + \tau_2] - j2\pi\nu_2(t_1 - t_2 - \tau_1)} \right].
\end{aligned}
\tag{40}
$$

Making the change of variables:

$$
\begin{aligned}
t &= t_1 \\
\tau &= t_1 - t_2
\end{aligned}
\tag{41}
$$

we can simply somewhat to obtain

$$
\begin{aligned}
K_{\widehat{S}_x}(f_1, f_2 : T) = \int_{-\infty}^{\infty} d\tau_1 \int_{-\infty}^{\infty} d\tau_2 \; w(\tau_1) w(\tau_2) e^{-j2\pi f_1 \tau_1} e^{-j2\pi f_2 \tau_2} \int_{-\infty}^{\infty} d\nu_1 \int_{-\infty}^{\infty} d\nu_2 S_x(\nu_1) S_x(\nu_2) \\
\times \int_{-T/2}^{T/2} dt \int_{-T}^{T} d\tau \frac{1}{T^2} \left[e^{j2\pi\tau(\nu_1 - \nu_2)} e^{j2\pi\nu_2(\tau_1 - \tau_2)} + e^{j2\pi\tau(\nu_1 - \nu_2)} e^{j2\pi(\nu_1\tau_2 + \nu_2\tau_1)} \right].
\end{aligned}
$$

Noting the integration

$$
\frac{1}{T^2} \int_{-T/2}^{T/2} dt \int_{-T}^{T} e^{j2\pi\tau(\nu_1 - \nu_2)} d\tau = 2\mathrm{sinc}\left[2\pi T(\nu_1 - \nu_2)\right] \simeq \frac{1}{T} \delta(\nu_1 - \nu_2), \tag{42}
$$

where the approximation of the sinc function as an impulse of the same area is made under the assumption that $S_x(f)$ is locally smooth relative to the window bandwidth B_w. Using this approximation and the sifting property of the impulse function we obtain

$$
\begin{aligned}
K_{\widehat{S}_x}(f_1, f_2 : T) \simeq \int_{-\infty}^{\infty} \frac{S_x^2(\nu)}{T} d\nu \int_{-\infty}^{\infty} w(\tau_1) e^{-j2\pi(f_1 - \nu)\tau_1} d\tau_1 \\
\times \int_{-\infty}^{\infty} \left[w(\tau_2) e^{-j2\pi(f_2 - \nu)\tau_2} + w(\tau_2) e^{-j2\pi(f_2 + \nu)\tau_2} \right] d\tau_2.
\end{aligned}
\tag{43}
$$

Recognizing the integrals in τ_i as Fourier transforms we obtain

$$
K_{\widehat{S}_x}(f_1, f_2 : T) \simeq \frac{1}{T} \int_{-\infty}^{\infty} S_x^2(\nu) W(f_1 - \nu) \left[W(f_2 - \nu) + W(f_2 + \nu)\right] d\nu. \tag{44}
$$

Making the change of variable $\nu \to f_1 - \nu$ and recalling that $W(f) = W(-f)$, the covariance function for the BT PSD estimate can be written in a form analogous to that obtained for the covariance function of the estimated autocorrelation function:

$$\boxed{K_{\widehat{S}_x}(f_1, f_2 : T) \simeq \frac{1}{T}\int_{-\infty}^{\infty} S_x^2(f_1-\nu)W(\nu)\left\{W[\nu-(f_1-f_2)] + W[\nu-(f_1+f_2)]\right\} d\nu.} \quad (45)$$

Clearly, the covariance of BT PSD estimates are functions of $W(f)$ and shifted version of $W(f)$ involving the difference $(f_1 - f_2)$ and the sum $(f_1 + f_2)$.

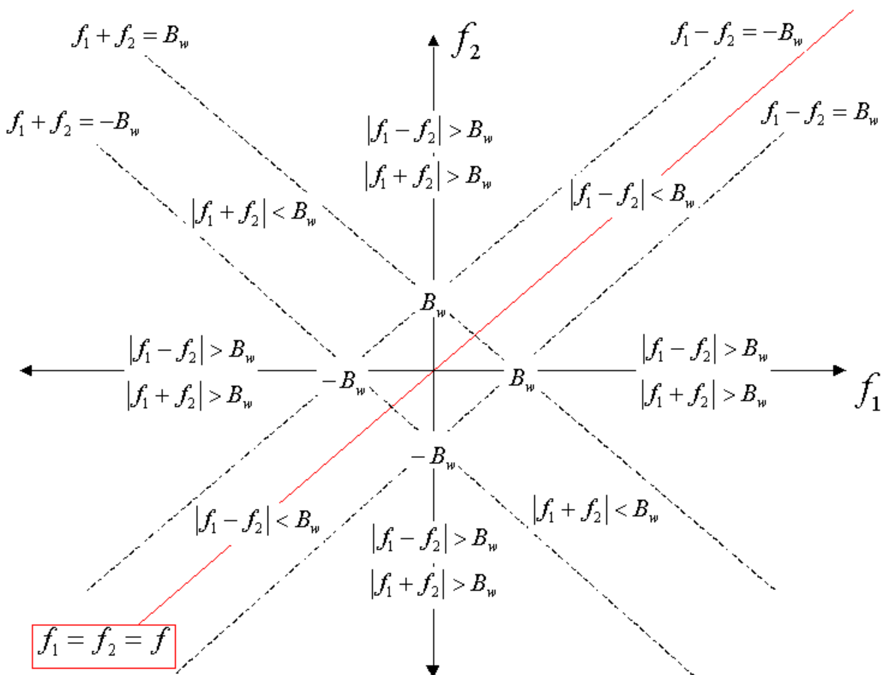

Figure 3: f_1, f_2–plane

- The assumption of locally smooth PSD leads to $S_x^2(f-\nu)W(\nu) \simeq S_x^2(f)W(\nu)$, from which we obtain the approximations (see figure 3):

$$K_{\widehat{S}_x}(f_1, f_2 : T) \simeq \begin{cases} 0, \text{ if } |f_1 \pm f_2| > B_w \text{ and} \\ \dfrac{S_x^2(f_1)}{T}\displaystyle\int_{-\infty}^{\infty} W(\nu)W[\nu-(f_1-f_2)]d\nu, \text{ if } |f_1 - f_2| < B_w \text{ and} \quad |f_1 + f_2| > B_w \\ \dfrac{S_x^2(f_1)}{T}\displaystyle\int_{-\infty}^{\infty} W(\nu)\left\{W[\nu-(f_1-f_2)] + W[\nu-(f_1+f_2)]\right\} d\nu, \quad |f_1 \pm f_2| < B_w \end{cases}.$$

- The variance of the BT PSD estimate is given by

$$\boxed{\sigma^2_{\widehat{S}_x}(f : T) = K_{\widehat{S}_x}(f, f : T) \simeq \frac{1}{T}\int_{-\infty}^{\infty} S_x^2(f-\nu)\left[W^2(\nu) + W(\nu)W(\nu-2f)\right] d\nu,} \quad (46)$$

and we similarly approximate it as

$$\sigma^2_{\widehat{S}_x}(f:T) \simeq \begin{cases} \dfrac{S_x^2(f)}{T}\displaystyle\int_{-\infty}^{\infty} W^2(\nu)d\nu = \dfrac{S_x^2(f)}{T}\displaystyle\int_{-\infty}^{\infty} w^2(\tau)d\tau, & |f| > B_w/2 \\ \dfrac{S_x^2(f)}{T}\displaystyle\int_{-\infty}^{\infty} W^2(\nu) + W(\nu)W(\nu-2f)d\nu, & |f| < B_w/2 \end{cases}.$$

- Define the following <u>*Window Factor:*</u>

$$\boxed{C_w \triangleq \frac{1}{M}\int_{-\infty}^{\infty} W^2(\nu)d\nu = \frac{1}{M}\int_{-\infty}^{\infty} w^2(\tau)d\tau} \tag{47}$$

- The normalized/standardized variance is given by

$$\sigma^2 = \frac{\sigma^2_{\widehat{S}_x}(f:T)}{S_x^2(f)} \simeq \frac{1}{T}\int_{-\infty}^{\infty} W^2(\nu) + W(\nu)W(\nu-2f)d\nu. \tag{48}$$

  Note that it is exclusively a function of parameters we can control/design. At dc ($f=0$) we have

$$\sigma^2 = \frac{\sigma^2_{\widehat{S}_x}(0:T)}{S_x^2(0)} \simeq 2\frac{MC_w}{T}. \tag{49}$$

  and removed from dc ($|f| > B_w/2$) we have

$$\sigma^2 = \frac{\sigma^2_{\widehat{S}_x}(f:T)}{S_x^2(f)} \simeq \frac{1}{T}\int_{-\infty}^{\infty} W^2(\nu)d\nu = \frac{MC_w}{T}. \tag{50}$$

  Hence, the window factor C_w and the length of the window M affect the variance/stability of our PSD estimate. Note that one can decrease the variance of the spectral estimate by reducing the window length M or choosing a window with smaller window factor C_w.

- Note the trade-off in the bias and variance of the BT PSD estimate $\widehat{S}_x(f:T)$.
  - Increasing the window duration M or the window factor C_w reduces the bias but increases the variance.
  - Decreasing the window duration M or the window factor C_w increases the bias, but reduces the variance.

- Often in spectral analysis it is of interest to resolve (identify as separate) closely spaced tones in the PSD. To this end we define the <u>*resolution factor*</u> as

$$\begin{aligned} \mathrm{RES}(f_1,f_2) &\triangleq \frac{K_{\widehat{S}_x}(f_1,f_2:T)}{\left(\frac{M}{T}\right)S_x^2(f_1)} \\ &= \frac{1}{M}\int_{-\infty}^{\infty} W(\nu)W[\nu-(f_1-f_2)] + W(\nu)W[\nu-(f_1+f_2)]d\nu. \end{aligned} \tag{51}$$

Away from the origin we obtain

$$\mathrm{RES}(f_1, f_2) \simeq \frac{1}{M}\int_{-\infty}^{\infty} W(\nu)W[\nu - (f_1 - f_2)]d\nu \overset{\text{“ideal”}}{\approx} C_w \cdot \delta_0(f_1 - f_2). \qquad (52)$$

where $\delta_0(f = 0) = 1$ and $\delta_0(f \neq 0) = 0$. Ideally we would like $\mathrm{RES}(f_1, f_2)$ to approximate this function.

- We say that tones at frequencies f_1 and f_2 are resolvable if $\mathrm{RES}(f_1, f_2) \simeq 0$. Clearly, the window function bandwidth B_w is the significant design parameter determining achievable resolution.
  - Thus, to improve resolution we want to reduce $B_w \simeq 1/M$, or equivalently increase the window length M (recall that larger M also reduces bias of BT PSD estimate, but increases its variance).
- Given a window function of bandwidth B_w:
  - If $|f_1 - f_2| < B_w$, then tones are unresolvable.
  - If $|f_1 - f_2| \geq B_w$, then tones are resolvable.
- Sidelobe leakage can be important when spectrum consists of closely spaced tones of differing strengths. The effect of sidelobe leakage can be adjusted/designed through proper choice of window function. See Art's notes example 4 and F. Harris paper Table I for tabulated values of peak sidelobe levels of various windowing functions, as well as mainlobe widths.

Fourier Method / Direct Method

- The BT Method (Indirect Method) estimated the process autocorrelation function from the data record, and subsequently windowed and transformed to the frequency domain to obtain the spectral estimate. In the Direct Method the data record is immediately windowed and transformed to the frequency domain. The spectral estimate is obtained by local averaging in the frequency domain.

- *It is emphasized that the assumptions on the windows used for the Direct Method will differ from those we made for windows applied using the BT/Indirect Method. In addition our definition of the window factor C_w will also differ. More said later.*

- Consider the transform of the windowed data record:

$$X_w(f:T) \triangleq \int_{-\infty}^{\infty} w(t)x(t)e^{-j2\pi ft}dt. \tag{53}$$

It is interesting to interpret this data from a filtering perspective. Let $h(t) = w(-t)e^{j2\pi f_0 t}$, and note that

$$y(t)|_{t=0} = x(t) * h(t)|_{t=0} = \int_{-\infty}^{\infty} w(\tau)x(\tau)e^{-j2\pi f_0\tau}d\tau = X_w(f_0:T) \tag{54}$$

i.e. the windowed data can be interpreted as the output of the filter $h(t)$ sampled at time zero. Consider, for example, $w(t) = 1/T$ for $|t| \leq T/2$. The frequency response is given by

$$H(f) = \frac{1}{T}\int_{-T/2}^{T/2} e^{j2\pi(f_0-f)t} = \text{sinc}\left[2\pi(f_0-f)\frac{T}{2}\right], \tag{55}$$

a bandpass filter with center frequency f_0 and 3dB bandwidth $B_w \simeq 1/T$.

- Consider figure 4, which ilustrates that disjoint frequencies of a random process are uncorrelated. In light of the filtering interpretation of the windowed data, note that frequencies separated by at least $B_w \simeq 1/T$ are uncorrelated.

- Assuming that the true PSD $S_x(f)$ is a relatively smooth function of frequency it is sensible to define the spectral estimate

$$\boxed{\widehat{S}_x(f:T) = \sum_{n=-N}^{N} h_n \left| X_w\left(f + \frac{n}{T}:T\right)\right|^2,} \tag{56}$$

where the weights h_n allow some flexibility in the averaging. It should be clear why this estimator is often termed the *smoothed periodogram.*

- Averaging correlated samples does improve the statistical stability of estimate, and can at times worsen it. Thus, averaging samples any closer than $1/T$ is not considered.

- It is desired to examine the bias and variance of this spectral estimate $\widehat{S}_x(f:T)$. Several related statistics, however, must be found first. Note that $X_w(f:T)$ has zero mean

$$E\{X_w(f:T)\} = \int_{-\infty}^{\infty} w(t)E\{x(t)\}e^{-j2\pi ft}dt = 0. \tag{57}$$

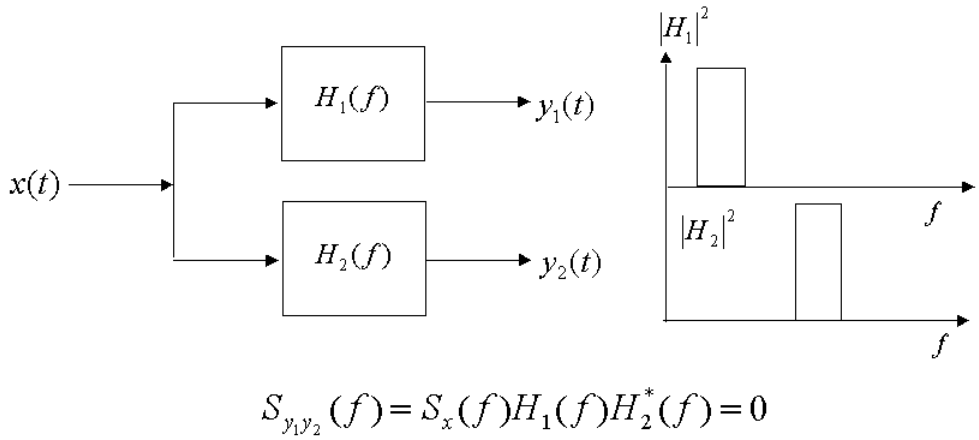

Figure 4: Disjoint frequencies are uncorrelated

Its covariance is given by

$$\begin{aligned}
K_{X_w}(f_1, f_2 : T) &\triangleq E\left[X_w(f_1 : T)X_w^*(f_2 : T)\right] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[x(t_1)x^*(t_2)]w(t_1)w^*(t_2)e^{-j2\pi(f_1t_1-f_2t_2)}dt_1dt_2 \qquad (58)\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_x(t_1-t_2)w(t_1)w^*(t_2)e^{-j2\pi(f_1t_1-f_2t_2)}dt_1dt_2.
\end{aligned}$$

To simplify, make the change of variables

$$\begin{aligned}
t &= t_1 \\
\tau &= t_1 - t_2.
\end{aligned} \qquad (59)$$

This leads to

$$\begin{aligned}
K_{X_w}(f_1,f_2:T) &= \int_{-\infty}^{\infty} w(t)\left[\int_{-\infty}^{\infty} R_x(\tau)w^*(t-\tau)e^{-j2\pi f_2\tau}d\tau\right]e^{-j2\pi f_1 t+j2\pi f_2 t}dt \\
&= \int_{-\infty}^{\infty} w(t)\left[\int_{-\infty}^{\infty} S_x(\nu)W^*(f_2-\nu)e^{-j2\pi(f_2-\nu)t}d\nu\right]e^{-j2\pi f_1 t+j2\pi f_2 t}dt \\
&= \int_{-\infty}^{\infty} S_x(\nu)W^*(f_2-\nu)d\nu\int_{-\infty}^{\infty} w(t)e^{-j2\pi(f_1-\nu)t}dt \qquad (60) \\
&= \int_{-\infty}^{\infty} S_x(\nu)W(f_1-\nu)W^*(f_2-\nu)d\nu \\
&= \int_{-\infty}^{\infty} S_x(f_1-\nu)W(\nu)W^*[\nu-(f_1-f_2)]d\nu.
\end{aligned}$$

Using familiar arguments we obtain the approximation

$$K_{X_w}(f_1,f_2:T) \simeq \begin{cases} 0, & |f_1-f_2| > B_w \\ S_x(f_1)\int_{-\infty}^{\infty} W(\nu)W^*[\nu-(f_1-f_2)]d\nu, & |f_1-f_2| < B_w \end{cases}. \qquad (61)$$

- The variance of $X_w(f:T)$ is given by

$$\begin{aligned}
\sigma^2_{X_w}(f:T) &\triangleq K_{X_w}(f,f:T) \\
&= \int_{-\infty}^{\infty} S_x(f-\nu)|W(\nu)|^2 d\nu \simeq S_x(f)\int_{-\infty}^{\infty}|W(\nu)|^2 d\nu
\end{aligned} \qquad (62)$$

- The variance of $|X_w(f:T)|^2$ is by definition given by

$$K_{|X_w|^2}(f_1,f_2:T) \triangleq E\left\{|X_w(f_1:T)|^2|X_w(f_2:T)|^2\right\} - E\left\{|X_w(f_1:T)|^2\right\}E\left\{|X_w(f_2:T)|^2\right\}$$

or expanded

$$K_{|X_w|^2}(f_1,f_2:T) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dt_1dt_2dt_3dt_4 \;\; E\{x(t_1)x(t_2)x(t_3)x(t_4)\} \qquad (63)$$

$$w(t_1)w^*(t_2)w(t_3)w^*(t_4)e^{-j2\pi f_1(t_1-t_2)-j2\pi f_2(t_3-t_4)} - K_{X_w}(f_1,f_1:T)K_{X_w}(f_2,f_2:T).$$

Recall the moment formula for Gaussians

$$\begin{aligned}
E\{x(t_1)x(t_2)x(t_3)x(t_4)\} &= R_x(t_1-t_2)R_x(t_3-t_4) \qquad (64) \\
&+R_x(t_1-t_3)R_x(t_2-t_4)+R_x(t_1-t_4)R_x(t_2-t_3).
\end{aligned}$$

By comparing this to eq(58) we obtain

$$\begin{aligned}
K_{|X_w|^2}(f_1,f_2:T) &= K_{X_w}(f_1,f_1:T)K_{X_w}(f_2,f_2:T) + \qquad (65) \\
&K_{X_w}(f_1,f_2:T)K_{X_w}(-f_1,-f_2:T) + \\
&K_{X_w}(f_1,-f_2:T)K_{X_w}(-f_1,f_2:T) - K_{X_w}(f_1,f_1:T)K_{X_w}(f_2,f_2:T).
\end{aligned}$$

Noting that $K_{X_w}(-f_1, -f_2 : T) = K^*_{X_w}(f_1, f_2 : T)$, we obtain the useful result

$$\boxed{K_{|X_w|^2}(f_1, f_2 : T) = |K_{X_w}(f_1, f_2 : T)|^2 + K_{X_w}(f_1, -f_2 : T)K_{X_w}(-f_1, f_2 : T).} \tag{66}$$

- Recalling eq(56), note that the mean of the PSD estimate is given by

$$\begin{aligned} E\left\{\widehat{S}_x(f : T)\right\} &= \sum_{n=-N}^{N} h_n \cdot \sigma^2_{X_w}\left(f + \frac{n}{T} : T\right) \\ &\simeq \sum_{n=-N}^{N} h_n \cdot S_x\left(f + \frac{n}{T}\right) \int_{-\infty}^{\infty} |W(\nu)|^2 d\nu \end{aligned} \tag{67}$$

If we assume that the true PSD is locally smooth across the bandwidth $B_h \triangleq 2N/T$, *i.e.* $S_x(f + n/T) \approx S_x(f)$, then to obtain an unbiased estimator of the PSD requires that

$$\boxed{\int_{-\infty}^{\infty} |W(\nu)|^2 d\nu \cdot \sum_{n=-N}^{N} h_n = 1.} \tag{68}$$

- In our analysis of the BT/Indirect method of PSD estimation we adopted the window constraint that w(0) = 1. This guaranteed approximately unbiased spectral estimates where $S_x(f)$ is locally smooth relative to the window function bandwidth B_w. For the Fourier/Direct Method an approximately unbiased spectral estimate is obtained by satisfying eq(68). We do, however, for the Direct Method adopt the following unit area convention for the windowing function:

$$\int_{-\infty}^{\infty} w(t)dt = 1 = W(0). \tag{69}$$

- Note that if we excite the following linear time-invariant (LTI) system with white noise $n(t)$

$$n(t) \longrightarrow \boxed{W(f)} \longrightarrow y(t) \tag{70}$$

where $R_n(\tau) = N_0 \cdot \delta(\tau)$, then the average output power is given by

$$E\{|y(t)|^2\} = N_0 \int_{-\infty}^{\infty} |W(\nu)|^2 d\nu. \tag{71}$$

Following Harris' development (see paper), the equivalent noise bandwidth (ENBW) can be obtained from

$$B_n = \frac{N_0 \int_{-\infty}^{\infty} |W(\nu)|^2 d\nu}{W^2(0)} = \frac{N_0 \int_{-\infty}^{\infty} |W(\nu)|^2 d\nu}{\left[\int_{-\infty}^{\infty} w(\tau)d\tau\right]^2}. \tag{72}$$

Normalizing by $B_w N_0 \simeq N_0/T$ (noise power per sample) we obtain the ENBW

$$\boxed{\text{ENBW} = \frac{B_n}{B_w N_0} = T\int_{-\infty}^{\infty} |W(\nu)|^2 d\nu = T\int_{-\infty}^{\infty} |w(\tau)|^2 d\tau \stackrel{\triangle}{=} C_w.} \tag{73}$$

- Note above that we have redefined the *window factor* for the Direct Method. Harris has tabulated these for various window functions (see Table I), which is useful for design.
- Note that for unbiased spectral estimates we want to choose weights h_n such that

$$\sum_{n=-N}^{N} h_n = \frac{T}{C_w}. \tag{74}$$

- Concerning the covariance function of the spectral estimate eq(56), note that

$$K_{\widehat{S}_x}(f_1, f_2 : T) \stackrel{\triangle}{=} E\left\{\widehat{S}_x(f_1 : T)\widehat{S}_x(f_2 : T)\right\} - E\left\{\widehat{S}_x(f_1 : T)\right\} E\left\{\widehat{S}_x(f_2 : T)\right\}. \tag{75}$$

Using this definition it is straight forward to show that

$$K_{\widehat{S}_x}(f_1, f_2 : T) = \sum_{n=-N}^{N}\sum_{m=-N}^{N} h_n h_m K_{|X_w|^2}\left(f_1 + \frac{n}{T}, f_2 + \frac{m}{T} : T\right). \tag{76}$$

Recall from eq(66) that

$$K_{|X_w|^2}\left(f_1 + \frac{n}{T}, f_2 + \frac{m}{T} : T\right) = \left|K_{X_w}\left(f_1 + \frac{n}{T}, f_2 + \frac{m}{T} : T\right)\right|^2 + \tag{77}$$

$$K_{X_w}\left(f_1 + \frac{n}{T}, -f_2 - \frac{m}{T} : T\right) K_{X_w}\left(-f_1 - \frac{n}{T}, f_2 + \frac{m}{T} : T\right).$$

Making the familiar arguments note that

$$K_{\widehat{S}_x}(f_1, f_2 : T) \simeq 0 \ \text{ if } \ \left|f_1 - f_2 + \frac{2N}{T}\right| > B_w. \tag{78}$$

- The variance of the Direct Method spectral estimate is given by

$$\sigma^2_{\widehat{S}_x}(f : T) \stackrel{\triangle}{=} K_{\widehat{S}_x}(f, f : T) = \sum_{n=-N}^{N}\sum_{m=-N}^{N} h_n h_m K_{|X_w|^2}\left(f + \frac{n}{T}, f + \frac{m}{T} : T\right), \tag{79}$$

where

$$K_{|X_w|^2}\left(f + \frac{n}{T}, f + \frac{m}{T} : T\right) = \left|K_{X_w}\left(f + \frac{n}{T}, f + \frac{m}{T} : T\right)\right|^2 + \tag{80}$$

$$K_{X_w}\left(f + \frac{n}{T}, -f - \frac{m}{T} : T\right) K_{X_w}\left(-f - \frac{n}{T}, f + \frac{m}{T} : T\right).$$

- Note that if

$$\left|f+\frac{N}{T}\right| > B_w/2 \ \text{ and } \ B_h \triangleq \frac{2N}{T} < B_w, \tag{81}$$

then

$$K_{|X_w|^2}\left(f+\frac{n}{T}, f+\frac{m}{T} : T\right) \simeq \left|K_{X_w}\left(f+\frac{n}{T}, f+\frac{m}{T} : T\right)\right|^2 \tag{82}$$

and therefore

$$\sigma^2_{\widehat{S}_x}(f:T) \simeq \sum_{n=-N}^{N}\sum_{m=-N}^{N} h_n h_m |\rho_{nm}(f)|^2 = \mathbf{h}^T\mathbf{R}_\rho\mathbf{h} \tag{83}$$

where

$$\begin{aligned} \rho_{nm}(f) &\triangleq K_{X_w}\left(f+\frac{n}{T}, f+\frac{m}{T} : T\right) \\ &= S_x(f)\int_{-\infty}^{\infty} W(\nu)W^*\left[\nu - \left(\frac{n-m}{T}\right)\right] d\nu \end{aligned} \tag{84}$$

and $[\mathbf{R}_\rho]_{n,m} \triangleq |\rho_{nm}(f)|^2$.

If $S_x(f)$ is spectrally smooth over $B_h = 2N/T$ and $B_w \simeq 1/T$, then

$$\rho_{nm}(f) = \begin{cases} \dfrac{S_x(f)\cdot C_w}{T}, & n = m \\ 0, & n \neq m. \end{cases} \tag{85}$$

Note on Confidence Intervals

- Confidence intervals are useful for communicating the possible nearness/closeness of an estimated parameter to the true value of the parameter.
- Recall that if $\tilde{\mathbf{z}} \sim CN_B(\mathbf{0}, \mathbf{I}_B)$, that is complex circular Gaussian, then it follows that

$$||\tilde{\mathbf{z}}||^2 = \sum_{i=1}^{B} |\tilde{z}_i|^2 \stackrel{d}{=} \chi_B^2, \tag{86}$$

where χ_B^2 is known as a *complex chi-squared* statistic. It has a probability density function given by

$$f_{\chi_B^2}(a) = a^{N-1} e^{-a}/(N-1)! \qquad a \geq 0. \tag{87}$$

- Recall that the windowed tranformed data record is given by

$$X_w(f : T) \triangleq \int_{-\infty}^{\infty} w(t)x(t)e^{-j2\pi ft}dt, \tag{88}$$

and the resulting spectral estimate by

$$\boxed{\widehat{S}_x(f : T) = \sum_{n=-N}^{N} h_n \left| X_w\left(f + \frac{n}{T} : T\right)\right|^2.} \tag{89}$$

If we assume that $x(t)$ is a zero mean Gaussian process, then it follows that

$$X_w(f : T) \sim CN_1\left[0, \sigma_{X_w}^2(f : T)\right], \quad \text{and} \quad |X_w(f : T)|^2 \sim \sigma_{X_w}^2(f : T) \cdot \chi_1^2 \tag{90}$$

Recall from eq(61) that samples of $X_w(f : T)$ at frequency separated by more than $B_w \simeq 1/T$ are uncorrelated, and hence independent. Thus, we argue that

$$\widehat{S}_x(f : T) \stackrel{d}{=} \sum_{n=-N}^{N} h_n \cdot \sigma_{X_w}^2\left(f + \frac{n}{T} : T\right) \cdot {}_n\chi_1^2. \tag{91}$$

Assuming $S_x(f)$ is spectrally smooth we obtain

$$\boxed{\widehat{S}_x(f : T) \stackrel{d}{=} \frac{S_x(f)C_w}{T} \sum_{n=-N}^{N} h_n \cdot {}_n\chi_1^2.} \tag{92}$$

- Confidence intervals can be obtained from this stochastic representation.
- Note that if $h_n = h = T/[(2N+1)C_w]$, then

$$\frac{\widehat{S}_x(f : T)}{S_x(f)} \stackrel{d}{=} \chi_{(2N+1)}^2. \tag{93}$$

Direct Method: Segmented

- The segmented Direct Method divides the data record $x_T(t)$ into L segments of equal duration T_L. Each segment is processed separately, then combined (averaged) to generate the resulting spectral estimate.

$$\widehat{S}_x(f:T) = \sum_{l=1}^{L} \sum_{n=-N}^{N} h_n \left| X_w \left(f + \frac{n}{T_L} : T_L \right) \right|^2 . \tag{94}$$

- Note that

$$E\{\widehat{S}_x(f:T)\} = S_x(f) \cdot \frac{L \cdot C_w}{T_L} \sum_{n=-N}^{N} h_n . \tag{95}$$

- Hence, for asymptotically unbiased estimates we require

$$\sum_{n=-N}^{N} h_n = \frac{T_L}{L \cdot C_w} . \tag{96}$$

- Can show that variance of $\widehat{S}_x(f:T)$ has similar form as unsegmented Direct Method with

$$\rho_{nm}^{kl}(f) \simeq S_x(f) \int_{-\infty}^{\infty} W(\nu) W^* \left[\nu - \left(\frac{n-m}{T_L} \right) \right] e^{-j2\pi(T_k - T_L)\nu} d\nu . \tag{97}$$