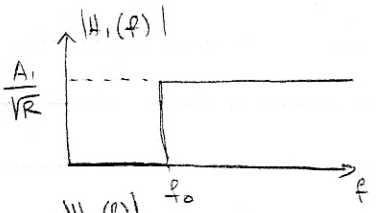
Problem set 2 solutions

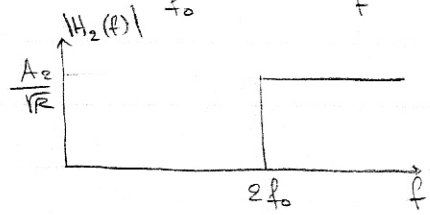
Problem 1

a) **Gain:**

$$H_1(f) = \begin{cases} \frac{A_1}{\sqrt{R}}, & f > f_0 \\ 0, & f \le f_0 \end{cases}$$



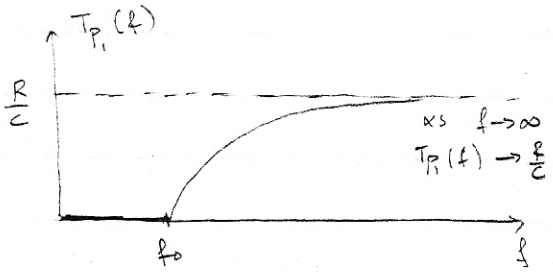
$$H_2(f) = \begin{cases} \frac{A_2}{\sqrt{R}}, & f > 2f_0 \\ 0, & f \le 2f_0 \end{cases}$$

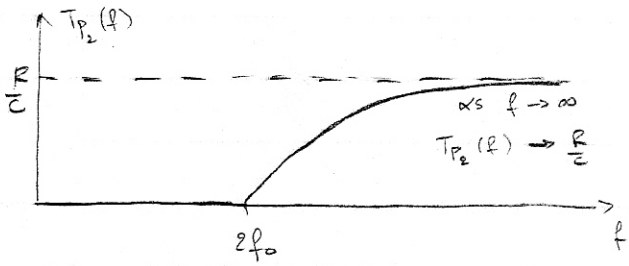


Phase delay:

$$T_{p_1}(f) = -\frac{1}{2\pi f} \measuredangle H_1(f) = \frac{R\,\frac{2\pi}{c}\sqrt{f^2 - f_0^2}}{2\pi f} = \begin{cases} \frac{R}{c}\,\frac{\sqrt{f^2 - f_0^2}}{f}, & f > f_0 \\ 0, & f \le f_0 \end{cases}$$

$$T_{p_2}(f) = -\frac{1}{2\pi f} \measuredangle H_2(f) = \frac{R\,\frac{2\pi}{c}\sqrt{f^2 - 4f_0^2}}{2\pi f} = \begin{cases} \frac{R}{c}\,\frac{\sqrt{f^2 - 4f_0^2}}{f}, & f > 2f_0 \\ 0, & f \le 2f_0 \end{cases}$$

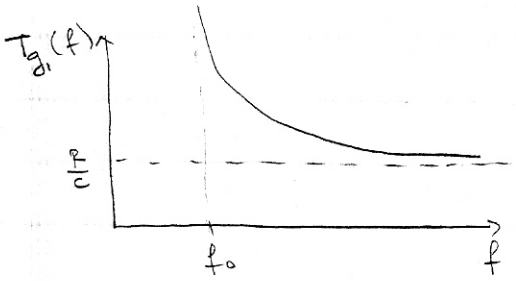


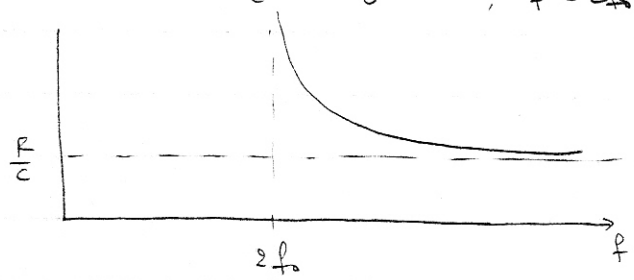


Group delay:

$$T_{g_1}(f) = -\frac{1}{2\pi}\frac{d}{df} \measuredangle H_1(f) = \frac{2\pi R}{2\pi c}\,\frac{2f}{2\sqrt{f^2 - f_0^2}} = \begin{cases} \frac{R}{c}\,\frac{f}{\sqrt{f^2 - f_0^2}}, & f > f_0 \\ 0, & f \le f_0 \end{cases}$$

$$T_{g_2}(f) = -\frac{1}{2\pi}\frac{d\,\angle H_2(f)}{df} = \frac{2\pi R}{2\pi}\frac{2f}{\sqrt{f^2-4f_0^2}} = \begin{cases} \frac{R}{c}\frac{1}{\sqrt{f^2-4f_0^2}} & f > 2f_0 \\ 0, & f \le 2f_0 \end{cases}$$





b) $H(f) = H_1(f) + H_2(f)$ → you may assume that the transmitted signal travels through 2 paths and gets added at the receiver.

The complex envelope of $x(t)$ is:

$$x(t) = e^{-(wt)^2}\cos(2\pi 4f_0 t) = 2\,\mathrm{Re}\left\{\underbrace{\frac{1}{2}e^{-(wt)^2}}_{\tilde{x}(t)} e^{j2\pi \underbrace{4f_0}_{\text{carrier freq.}} t}\right\}$$

$$\Rightarrow \tilde{x}(t) = \frac{1}{2}e^{-(wt)^2}$$

and the output is:

$$y(t) = 2\,\mathrm{Re}\left\{\tilde{x}(t - T_{g_1}(f))\,|H_1(4f_0)|\,e^{j2\pi 4f_0(t - T_{p_1}(4f_0))}\right\} +$$

$$2\,\mathrm{Re}\left\{\tilde{x}(t - T_{g_2}(f))\,|H_2(4f_0)|\,e^{j2\pi 4f_0(t - T_{p_2}(4f_0))}\right\}$$

$$= \frac{A_1}{\sqrt{R}}e^{-w^2\left(t - \frac{4}{\sqrt{15}}\frac{R}{c}\right)}\cos\left[8\pi f_0\left(t - \frac{R}{c}\frac{\sqrt{15}}{4}\right)\right] +$$

$$\frac{A_2}{\sqrt{R}}e^{-w^2\left(t - \frac{4}{\sqrt{12}}\frac{R}{c}\right)}\cos\left[8\pi f_0\left(t - \frac{R}{c}\frac{\sqrt{12}}{4}\right)\right]$$

c) In order the two components to be separable in time their group delays must be separated by $\frac{1}{W}$ thus:

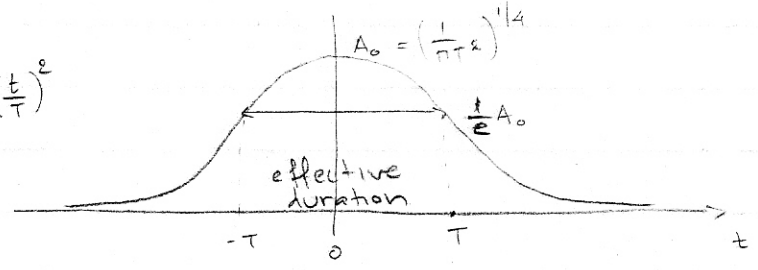
$$\frac{4R}{\sqrt{12}\,c} - \frac{4R}{\sqrt{15}\,c} > \frac{1}{W} \Rightarrow \frac{4R}{c}\,\frac{\sqrt{15}-\sqrt{12}}{\sqrt{15\cdot 12}} > \frac{1}{W} \Rightarrow$$

$$\boxed{R > \frac{c\sqrt{15\cdot 12}}{4W\left(\sqrt{15}-\sqrt{12}\right)}}$$

Problem 2

Let's first calculate the time-freq. autocorrelation function of the simple Gaussian pulse (without the chirp)

$$\tilde{f}(t) = \left(\frac{1}{2\pi T^2}\right)^{1/4} e^{-\frac{1}{4}\left(\frac{t}{T}\right)^2}$$
$$-\infty \le t \le \infty$$



$$\Phi_f(\Delta T, \Delta f) = \int_{-\infty}^{\infty} \tilde{f}\left(t + \frac{\Delta T}{2}\right) \tilde{f}^*\left(t - \frac{\Delta T}{2}\right) e^{-j2\pi \Delta f t}\, dt =$$

$$= \int_{-\infty}^{\infty} \left(\frac{1}{2\pi T^2}\right)^{1/2} e^{-\frac{1}{4}\left(\frac{t+\frac{\Delta T}{2}}{T}\right)^2} e^{-\frac{1}{4}\left(\frac{t-\frac{\Delta T}{2}}{T}\right)^2} e^{-j2\pi \Delta f t} =$$

$$= \left(\frac{1}{2\pi T^2}\right)^{1/2} \int_{-\infty}^{\infty} e^{-\left(\frac{t^2 + \frac{\Delta T^2}{4} + t\Delta T + t^2 + \frac{\Delta T^2}{4} - t\Delta T}{4T^2}\right)} e^{-j2\pi \Delta f t}\, dt =$$

$$= \left(\frac{1}{2\pi T^2}\right)^{1/2} \int_{-\infty}^{\infty} e^{-\left(\frac{2t^2 + \Delta T^2/2}{4T^2}\right)} e^{-j2\pi \Delta f t}\, dt$$

$$= \int_{-\infty}^{\infty} \left(\frac{1}{2\pi T^2}\right)^{1/2} \cdot e^{-\frac{1}{2}\left(\frac{t}{T}\right)^2} \cdot e^{-j2\pi \Delta f t}\, dt \; e^{-\Delta T^2/8T^2}$$

using the hint:

$$\left(\frac{1}{2\pi T^2}\right)^{1/2} e^{-\frac{1}{2}\left(\frac{t}{T}\right)^2} \xleftrightarrow{\mathcal{F}} e^{-2(\pi f T)^2}$$

$$\Phi_f(\Delta T, \Delta f) = e^{-2(\pi \Delta f T)^2} e^{-\Delta T^2/8T^2} = e^{-\frac{1}{8}\left(\frac{\Delta T^2}{T^2} + (4\pi \Delta f T)^2\right)}$$

Hence the ambiguity function is

$$\Theta_f(\Delta T, \Delta f) = \left| \Phi_f(\Delta T, \Delta f) \right|^2 = \left| e^{-\frac{1}{8}\left(\frac{\Delta T^2}{T^2} + (4\pi \Delta f T)^2 \right)} \right|^2$$

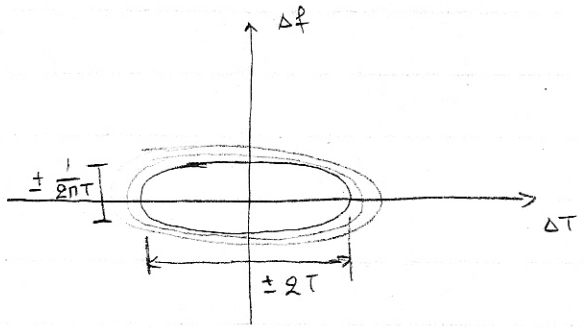
$$= e^{-\frac{1}{4}\left(\frac{\Delta T^2}{T^2} + (4\pi T \Delta f)^2 \right)}$$

Looking at the exponent $\frac{\Delta T^2}{T^2} + (4\pi T \Delta f)^2 = c$ we note that the contours of equal magnitude are ellipses. Define accuracy at the point where Θ_f falls to e^{-1}. Measuring the accuracy in Doppler axis we set

$$\Delta T = 0 \text{ and } -\frac{1}{4}(4\pi T \Delta f)^2 = -1 \rightarrow \Delta f = \pm \frac{2}{4\pi T} = \pm \frac{1}{2\pi T}$$

Measuring the accuracy in range axis we set

$$\Delta f = 0 \text{ and } -\frac{1}{4}\frac{\Delta T^2}{T^2} = -1 \rightarrow \Delta T = \pm 2T$$



So we have one parameter i.e. the pulse duration, to control both range and Doppler accuracy.

In class we said that the scattering function $S(\tau, f_d)$ shows how the power of the target is distributed across delay (range) and Doppler (speed). For a single reflector with true delay $\bar{\tau}$ and Doppler $\bar{f}_d$ the scattering function is a 2-D impulse

$S(\tau, f_d) = \delta(\tau - \bar{\tau}, f_d - \bar{f}_d)$ or $S(\Delta T, \Delta f) = \delta(\Delta T, \Delta f)$

if we shift it to the origin of the range-Doppler plane.

In our case we have two reflectors so the scattering function is the sum of two impulses at $(0,0)$ and $(T_s, \frac{1}{T_s})$ respectively. The output of the correlation receiver in general is:

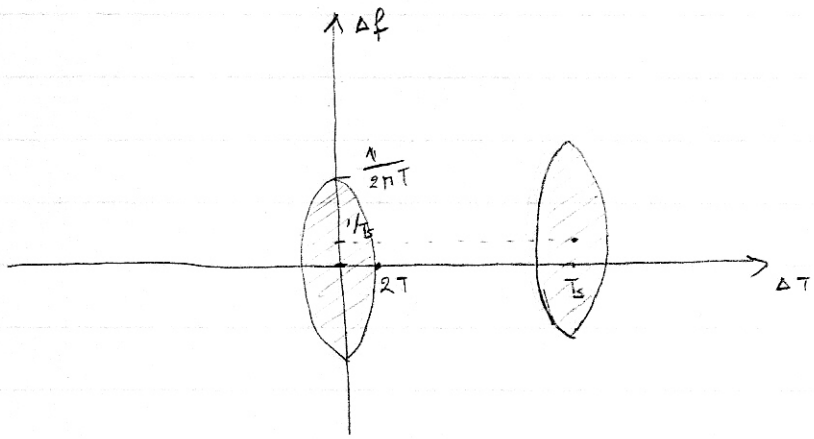
$$E\left[|\ell(\Delta T, \Delta f)|^2\right] = S(\Delta T, \Delta f) * \Theta_f(\Delta T, \Delta f)$$

$$\delta(\Delta T, \Delta f) + \delta(\Delta T - T_s, \Delta f - \tfrac{1}{T_s}) \quad \text{2-D convolution}$$

$$= \Theta_f(\Delta T, \Delta f) + \Theta_f(\Delta T - T_s, \Delta f - \tfrac{1}{T_s})$$

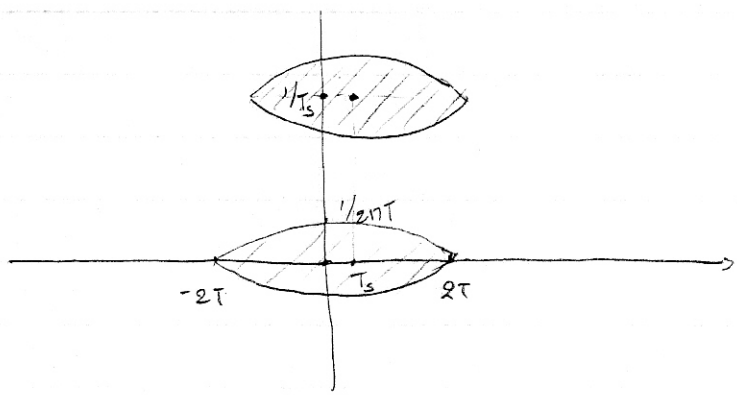
a) $T_s \gg T \Rightarrow \frac{1}{T} \gg \frac{1}{T_s}$

the output looks like:



note that we can resolve the two targets.

$T_s \ll T \Rightarrow \frac{1}{T_s} \gg \frac{1}{T}$



we can distinguish the two targets.

b) Now we have a Gaussian LFM pulse. We proved in class that

$$\Theta_{f_{LFM}}(\Delta T, \Delta f) = \Theta_{f}(\Delta T, \Delta f - \mu \Delta T)$$

$$= e^{-\frac{1}{4}\left(\frac{\Delta T^2}{T^2} + (4\pi T(\Delta f - \mu \Delta T))^2\right)}$$

Looking at the exponent we have:

$$\frac{\Delta T^2}{T^2} + 16\pi^2 T^2 \left(\Delta f^2 - 2\Delta f \mu \Delta T + \mu^2 \Delta T^2\right) =$$

$$= \Delta T^2 \left(\frac{1}{T^2} + 16\pi^2 T^2 \mu^2\right) - 32\pi^2 T^2 \mu \Delta f \Delta T + 16\pi^2 T^2 \Delta f^2$$

Define accuracy at the point where the magnitude of $\Theta_{f_{LFM}}$ falls to e^{-1}

Accuracy in Doppler: $\Delta T = 0 \Rightarrow -\frac{1}{4}\left(16\pi^2 T^2 \Delta f^2\right) = -1$

$$\Rightarrow \Delta f = \pm \frac{1}{2\pi T}$$

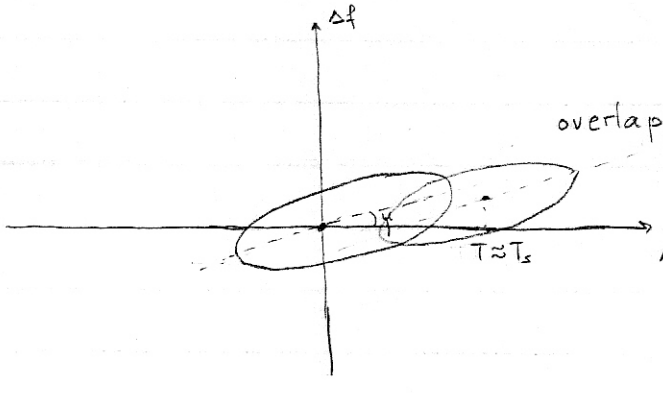
Accuracy in range: $\Delta f = 0 \Rightarrow -\frac{1}{4}\left(\Delta T^2 \left(\frac{1}{T^2} + 16\pi^2 T^2 \mu^2\right)\right) = -1$

in practice $\mu \cdot T \gg 1/T$ so

$\frac{1}{T^2} \ll 16\pi^2 T^2 \mu^2$ hence

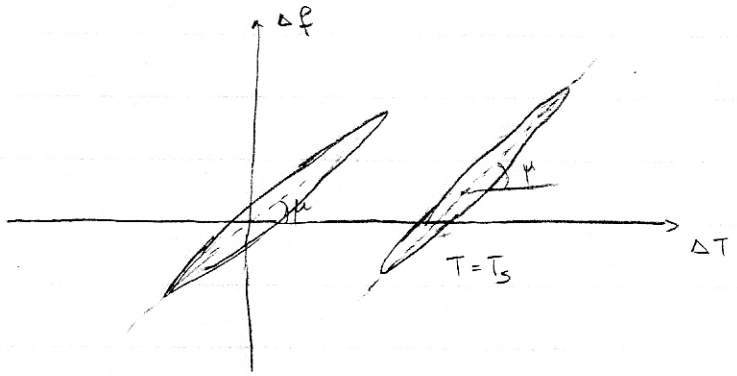
$$-\frac{1}{4}\Delta T^2 (4\pi T \mu)^2 \approx -1 \rightarrow \Delta T \simeq \pm \frac{1}{2\pi \mu T}$$

Note that by increasing T we get good accuracy in both Doppler and range. Range accuracy is achieved due to the "pulse compression property" of LFM signals.



Observe that for small μ the ambiguity functions overlap so we will be having a difficulty to resolve the targets even in the absence of noise.

big μ
$T \simeq T_s$

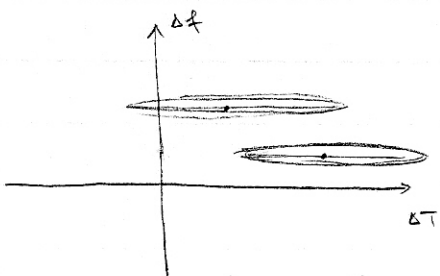


Observe that for big μ we can resolve the two targets.

c) range gating: we need a small pulse (large BW) in order to have a good range accuracy. But a chirp with long duration and big μ is effective due to its pulse compression property. Recall that range accuracy for chirps $\sim 1/2\pi\mu T$.

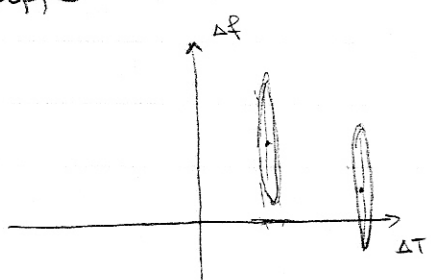
Doppler gating: we need a long pulse (small BW) for good Doppler accuracy. A chirp with small μ is fine in this case. Recall that Doppler accuracy is $\sim \frac{1}{T}$.

a priori knowledge in range



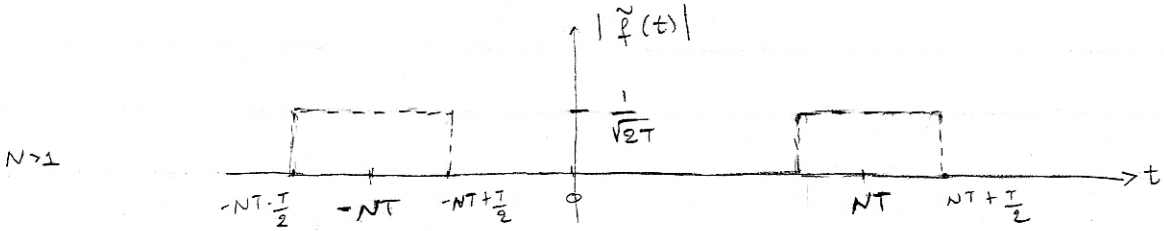
I want big T and small $\mu \simeq 0$

a priori knowledge in Doppler



I want big T and big μ.

Problem 3



a) $$\phi(\Delta T, \Delta f) = \int_{-\infty}^{\infty} \tilde{f}\left(t+\frac{\Delta T}{2}\right) \tilde{f}^*\left(t-\frac{\Delta T}{2}\right) e^{-j2\pi\Delta f t}\, dt$$

$$\Phi_f(0,\Delta f) = \int_{-\infty}^{\infty} \tilde{f}(t)\,\tilde{f}^*(t)\, e^{-j2\pi\Delta f t}\, dt = \int_{-\infty}^{\infty} |\tilde{f}(t)|^2 e^{-j2\pi\Delta f t}\, dt$$

$$= \frac{1}{2T}\int_{-NT-\frac{T}{2}}^{-NT+T/2} e^{-j2\pi\Delta f t}\, dt + \frac{1}{2T}\int_{NT-\frac{T}{2}}^{NT+T/2} e^{-j2\pi\Delta f t}\, dt =$$

$$= \frac{1}{2T}\left(e^{j2\pi NT\Delta f}\,\frac{\sin(\pi T\Delta f)}{\pi\Delta f} + e^{-j2\pi NT\Delta f}\,\frac{\sin(\pi\cdot T\Delta f)}{\pi\Delta f}\right)$$

$$= \operatorname{sinc}(\pi T\Delta f)\cdot\frac{e^{j2\pi NT\Delta f} + e^{-j2\pi NT\Delta f}}{2}$$

$$= \operatorname{sinc}(\pi T\Delta f)\cos(2\pi NT\Delta f) \Rightarrow$$

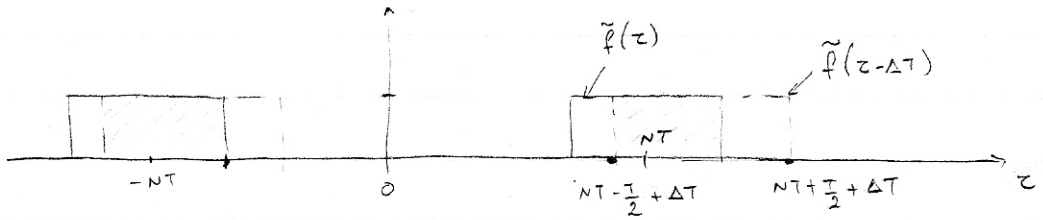
$$\Phi_f(0,\Delta f) = \operatorname{sinc}^2(\pi T\Delta f)\underbrace{\cos^2(2\pi NT\Delta f)}_{=\frac{1}{2}(1+\cos(4\pi NT\Delta f))}$$

$$\Phi_f(\Delta T, 0) = \int_{-\infty}^{\infty} \tilde{f}\left(t+\frac{\Delta T}{2}\right)\tilde{f}^*\left(t-\frac{\Delta T}{2}\right) dt \xrightarrow{t+\frac{\Delta T}{2}=\tau}$$

$$\int_{-\infty}^{\infty} \tilde{f}(\tau)\,\tilde{f}^*(\tau-\Delta T)\, d\tau$$

We have the following cases:

case 1: $T > \Delta T > 0$



$$\phi_f(\Delta T, 0) = \int_{-NT-\frac{T}{2}+\Delta T}^{-NT+T/2} \frac{1}{2T} e^{-j2\pi\frac{M}{T}\tau} e^{j2\pi\frac{M(\tau-\Delta T)}{T}} d\tau + \int_{NT-\frac{T}{2}+\Delta T}^{NT+\frac{T}{2}} \frac{1}{2T} e^{j2\pi\frac{M}{T}\tau} e^{-j2\pi\frac{M(\tau-\Delta T)}{T}} d\tau$$

$$= \frac{1}{2T} e^{-j2\pi M\frac{\Delta T}{T}} \int_{-NT-T/2+\Delta T}^{-NT+T/2} d\tau + \frac{1}{2T} e^{j2\pi M\frac{\Delta T}{T}} \int_{NT-T/2+\Delta T}^{NT+T/2} d\tau$$

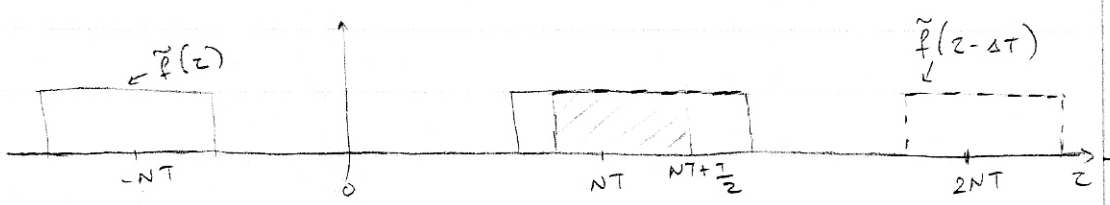
$$= \frac{(T-\Delta T)}{T} \frac{e^{j2\pi M\frac{\Delta T}{T}} + e^{-j2\pi M\frac{\Delta T}{T}}}{2} = \left(\frac{T-\Delta T}{T}\right)\cos\left(2\pi\frac{M}{T}\Delta T\right)$$

by symmetry when $-T < \Delta T < 0$ then

$$\phi_f(\Delta T, 0) = \left(\frac{T+\Delta T}{T}\right)\cos\left(-2\pi\frac{M}{T}\Delta T\right)$$

Hence $\phi(\Delta T, 0) = \left(\frac{T-|\Delta T|}{T}\right)\cos\left(2\pi\frac{M}{T}|\Delta T|\right)$, $|\Delta T| \le T$

case 2: $2NT < \Delta T < 2NT + T$



$$\phi_f(\Delta T, 0) = \int_{NT-\frac{T}{2}+\Delta T}^{NT+T/2} \frac{1}{2T} e^{j2\pi\frac{M\tau}{T}} e^{j2\pi M\frac{(\tau-\Delta T)}{T}} d\tau =$$

$$= \frac{1}{2T} e^{-j2\pi M\frac{\Delta T}{T}} \int_{NT-\frac{T}{2}+\Delta T}^{NT+\frac{T}{2}} e^{j2\pi\frac{2M}{T}\tau} d\tau$$

$$= \frac{1}{2T} e^{-j2\pi M\Delta T/T} \frac{\left[e^{j2\pi\frac{2M}{T}\left(NT+\frac{T}{2}\right)} - e^{j2\pi\frac{2M}{T}\left(-NT-\frac{T}{2}+\Delta T\right)}\right]}{j4\pi M/T}$$

$$= \frac{1}{2T} e^{-j2\pi \frac{M}{T}\Delta T} e^{j2\pi \frac{2M}{T}\left(NT + \frac{\Delta T}{2}\right)} \frac{e^{j2\pi \frac{2M}{T}\left(\frac{T}{2} - \frac{\Delta T}{2}\right)} - e^{-j2\pi \frac{2M}{T}\left(\frac{T}{2} - \frac{\Delta T}{2}\right)}}{j2\pi \frac{2M}{T}}$$

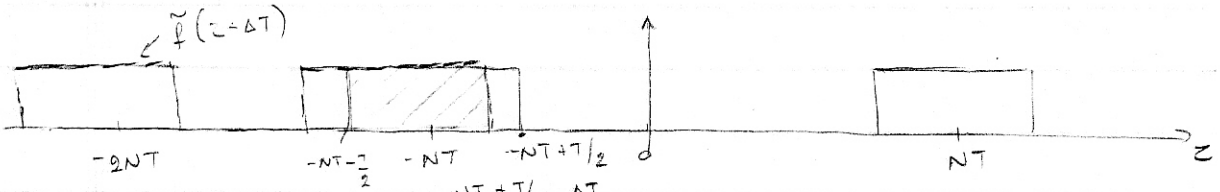
$$= \frac{1}{4\pi M} e^{-j2\pi \frac{M}{T}\left(\Delta T + 2NT + \Delta T\right)} \sin\left(2\pi \frac{2M}{T}\left(\frac{T}{2} - \frac{\Delta T}{2}\right)\right)$$

by symmetry when $2NT - T < \Delta T < 2NT$ we have the same result for $-\Delta T$ so

$$\Phi(\Delta T, 0) = \frac{1}{4\pi M} e^{-j2\pi \frac{M}{T}\left(2NT + 2|\Delta T|\right)} \sin\left[2\pi \frac{M}{T}\left(T - |\Delta T|\right)\right]$$

for $2NT - T \leq \Delta T \leq 2NT + T \Rightarrow |2NT - \Delta T| \leq T$

case 3: $-2NT - T < \Delta T < -2NT$



$$\Phi_{\tilde{f}}(\Delta T, 0) = \int_{-NT - T/2}^{-NT + T/2 - \Delta T} 1/2T\, e^{-j2\pi \frac{M}{T}\tau} e^{-j2\pi \frac{M}{T}(\tau + \Delta T)} d\tau$$

$$= \frac{1}{2T} e^{-j2\pi \frac{M}{T}\Delta T} \int_{-NT - \frac{T}{2}}^{-NT + \frac{T}{2} - \Delta T} e^{j2\pi \frac{2M}{T}\tau} d\tau =$$

$$= \frac{1}{2T} e^{-j2\pi \frac{M}{T}\Delta T} \left[\frac{e^{j2\pi \frac{2M}{T}\left(-NT + \frac{T}{2} - \Delta T\right)} - e^{j2\pi \frac{2M}{T}\left(-NT - T/2\right)}}{j4\pi M/T}\right]$$

$$= \frac{1}{4\pi M} e^{-j2\pi \frac{M}{T}\Delta T} e^{-j2\pi \frac{2M}{T}\left(NT + \frac{\Delta T}{2}\right)} \left[\frac{e^{j2\pi \frac{2M}{T}\left(\frac{T}{2} - \frac{\Delta T}{2}\right)} - e^{-j2\pi \frac{2M}{T}\left(\frac{T}{2} - \frac{\Delta T}{2}\right)}}{2j}\right]$$

$$= \frac{1}{4\pi M} e^{-j2\pi \frac{M}{T}\left(\Delta T + 2NT + \Delta T\right)} \sin\left(2\pi \frac{M}{T}\left(T - \Delta T\right)\right)$$

by symmetry when $-2NT < \Delta T < -2NT + T$
we have the same result for $-\Delta T$

so:

$$\Phi_{\tau}(\Delta T, 0) = \begin{cases} \left(1 - \frac{|\Delta T|}{T}\right) \cos\left(2\pi M \frac{|\Delta T|}{T}\right), & |\Delta T| \le T \\ \frac{1}{4\pi M} e^{-j 4\pi M \frac{|\Delta T|}{T}} \sin\left(2\pi M \left(1 - \frac{|\Delta T|}{T}\right)\right), & |2NT - \Delta T| \le T \\ \frac{1}{4\pi M} e^{-j 4\pi M \frac{|\Delta T|}{T}} \sin 2\pi M \left(1 - \frac{|\Delta T|}{T}\right), & |2NT + \Delta T| \le T \\ 0, & \text{o.w.} \end{cases}$$

$$\Theta_f(\Delta T, 0) = \begin{cases} \left(1 - \frac{|\Delta T|}{T}\right)^2 \cos^2\left(2\pi M \frac{|\Delta T|}{T}\right), & |\Delta T| \le T \\ \left(\frac{1}{4\pi M}\right)^2 \sin^2\left(2\pi M \left(1 - \frac{|\Delta T|}{T}\right)\right), & |2NT - \Delta T| \le T \\ \left(\frac{1}{4\pi M}\right)^2 \sin^2\left(2\pi M \left(1 - \frac{|\Delta T|}{T}\right)\right), & |2NT + \Delta T| \le T \\ 0, & \text{o.w.} \end{cases}$$

$$\Theta_f(0, \Delta f) = \operatorname{sinc}^2(\pi T \Delta f) \cos^2(2\pi N T \Delta f)$$

b) Doppler resolution: $\Theta_f(0, \Delta f) = \operatorname{sinc}^2(\pi T \Delta f) \cos^2(2\pi N T \Delta f)$

$= \operatorname{sinc}^2(\pi T \Delta f) \cdot \frac{1 + \cos(4\pi N T \Delta f)}{2}$

sinc^2: 1st zeros @ $\pm \frac{1}{T}$; $\frac{1+\cos}{2}$: 1st zeros @ $\pm \frac{1}{4NT}$

hence Doppler resolution is proportional to $\frac{1}{N}$ so we prefer N to be large.

Range resolution: main lobe resolution is $\pm \frac{T}{M}$ so large M helps to narrow the main lobe. Moreover, we have sidelobes @ $\pm 2NT$ which are proportional to $\frac{1}{M^2}$. Again, large M decreases the sidelobes. If we have a priori knowledge of the target location we can increase N so that the sidelobes fall outside our "known" target location.

Problem 4

a) $\tilde{f}_1(t) = \tilde{f}(t - T_x)$

$$\Phi_{f_1}(\Delta T, \Delta f) = \int_{-\infty}^{\infty} \tilde{f}_1\left(t + \frac{\Delta T}{2}\right) \tilde{f}_1^*\left(t - \frac{\Delta T}{2}\right) e^{-j2\pi \Delta f t}\, dt =$$

$$= \int_{-\infty}^{\infty} \tilde{f}\left(t - T_x + \frac{\Delta T}{2}\right) \tilde{f}^*\left(t - T_x - \frac{\Delta T}{2}\right) e^{-j2\pi \Delta f t}\, dt$$

change variables : $t - T_x = \tau$

$$\Phi_{f_1}(\Delta T, \Delta f) = \int_{-\infty}^{\infty} \tilde{f}\left(\tau + \frac{\Delta T}{2}\right) \tilde{f}^*\left(\tau - \frac{\Delta T}{2}\right) e^{-j2\pi \Delta f \tau}\, d\tau \; e^{-j2\pi \Delta f T_x}$$

$$\Rightarrow \Theta_{f_1}(\Delta T, \Delta f) = \underbrace{\left|e^{-j2\pi \Delta f T_x}\right|^2}_{1} \left| \int_{-\infty}^{+\infty} \tilde{f}\left(\tau + \frac{\Delta T}{2}\right) \tilde{f}^*\left(\tau - \frac{\Delta T}{2}\right) e^{-j2\pi \Delta f \tau}\, d\tau \right|^2$$

$= \Theta_f(\Delta T, \Delta f)$ so ambiguity function invariant to time shift

$$\tilde{f}_2(t) = \tilde{f}(t) e^{j2\pi f_x t}$$

$$\Phi_{f_2}(\Delta T, \Delta f) = \int_{-\infty}^{\infty} \tilde{f}_2\left(t + \frac{\Delta T}{2}\right) \tilde{f}_2^*\left(t - \frac{\Delta T}{2}\right) e^{-j2\pi \Delta f t}\, dt =$$

$$= \int_{-\infty}^{\infty} \tilde{f}\left(t + \frac{\Delta T}{2}\right) e^{j2\pi f_x t}\, e^{j2\pi f_x \frac{\Delta T}{2}}\, \tilde{f}^*\left(t - \frac{\Delta T}{2}\right) e^{-j2\pi f_x t}\, e^{j2\pi f_x \frac{\Delta T}{2}}\, e^{-j2\pi \Delta f t}\, dt$$

$$= \int_{-\infty}^{\infty} \tilde{f}\left(t + \frac{\Delta T}{2}\right) \tilde{f}^*\left(t - \frac{\Delta T}{2}\right) e^{-j2\pi \Delta f t}\, dt \; e^{j2\pi f_x \Delta T} \Rightarrow$$

$$\Theta_{f_2}(\Delta T, \Delta f) = \underbrace{\left|e^{j2\pi f_x \Delta T}\right|^2}_{1} \left| \int_{-\infty}^{\infty} \tilde{f}\left(t + \frac{\Delta T}{2}\right) \tilde{f}^*\left(t - \frac{\Delta T}{2}\right) e^{-j2\pi \Delta f t}\, dt \right|^2 =$$

$= \Theta_f(\Delta T, \Delta f)$ so ambiguity function invariant to freq shift.

We could combine the above in a single transform $\tilde{f}_3(t) = \tilde{f}(t - T_x) e^{j2\pi f_x t}$ and prove $\Theta_{f_3} = \Theta_f$

b) $$\Theta_f(-\Delta T, -\Delta f) = \left| \int_{-\infty}^{\infty} \tilde{f}\left(t-\tfrac{\Delta T}{2}\right) \tilde{f}^*\left(t+\tfrac{\Delta T}{2}\right) e^{j2\pi \Delta f t}\, dt \right|^2$$

$$= \left| \left(\int_{-\infty}^{\infty} \tilde{f}\left(t+\tfrac{\Delta T}{2}\right) \tilde{f}^*\left(t-\tfrac{\Delta T}{2}\right) e^{-j2\pi \Delta f t}\, dt \right)^* \right|^2 = \text{complex conjugates have the same magn.}$$

$$= \left| \int_{-\infty}^{\infty} \tilde{f}\left(t+\tfrac{\Delta T}{2}\right) \tilde{f}^*\left(t-\tfrac{\Delta T}{2}\right) e^{-j2\pi \Delta f t}\, dt \right|^2 = \Theta_f(\Delta T, \Delta f)$$

c) $$\Phi_f(\Delta T, \Delta f) = \int_{-\infty}^{\infty} \tilde{f}\left(t+\tfrac{\Delta T}{2}\right) \tilde{f}^*\left(t-\tfrac{\Delta T}{2}\right) e^{-j2\pi \Delta f t}\, dt =$$

$$= \text{F.T.}\left[\tilde{f}\left(t+\tfrac{\Delta T}{2}\right) \tilde{f}^*\left(t-\tfrac{\Delta T}{2}\right) \right] = \text{F.T.}\left[\tilde{f}\left(t+\tfrac{\Delta T}{2}\right) \right] * \text{F.T.}\left[\tilde{f}^*\left(t-\tfrac{\Delta T}{2}\right) \right]$$

$$\tilde{f}\left(t+\tfrac{\Delta T}{2}\right) \overset{F}{\longleftrightarrow} \tilde{F}(f)\, e^{j2\pi f \frac{\Delta T}{2}}$$

$$\tilde{f}^*\left(t-\tfrac{\Delta T}{2}\right) \overset{F}{\longleftrightarrow} \tilde{F}^*(-f)\, e^{-j2\pi f \Delta T/2}$$

thus $$\Phi_f(\Delta T, \Delta f) = \tilde{F}(f)\, e^{j2\pi f \frac{\Delta T}{2}} * \tilde{F}^*(-f)\, e^{-j2\pi f \Delta T/2} =$$

$$= \int_{-\infty}^{\infty} \tilde{F}(a)\, e^{j2\pi a \frac{\Delta T}{2}}\, \tilde{F}^*(a-\Delta f)\, e^{j2\pi (a-\Delta f)\Delta T/2}\, da =$$

$$= \int_{-\infty}^{\infty} \tilde{F}(a)\, \tilde{F}^*(a-\Delta f)\, e^{j2\pi a \Delta T}\, e^{-j2\pi \Delta f \Delta T/2}\, da$$

change variables $a \to a' + \frac{\Delta f}{2}$

$$= \int_{-\infty}^{\infty} \tilde{F}\left(a'+\tfrac{\Delta f}{2}\right) \tilde{F}^*\left(a'-\tfrac{\Delta f}{2}\right) e^{j2\pi a' \Delta T}\, da'$$

d) $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \underbrace{\Theta_f(\Delta T, \Delta f)}_{|\Phi_f(\Delta T, \Delta f)|^2 = \Phi_f(\Delta T, \Delta f)\, \Phi_f^*(\Delta T, \Delta f)} e^{-j2\pi u \Delta T}\, e^{j2\pi \tau \Delta f}\, d\Delta T\, d\Delta f =$$

$$\iint \int_{-\infty}^{\infty} \tilde{f}\left(t+\tfrac{\Delta T}{2}\right) \tilde{f}^*\left(t-\tfrac{\Delta T}{2}\right) e^{-j2\pi \Delta f t}\, dt \left[\int_{-\infty}^{\infty} \tilde{f}\left(\sigma+\tfrac{\Delta T}{2}\right) \tilde{f}^*\left(\sigma-\tfrac{\Delta T}{2}\right) e^{-j2\pi \Delta f \sigma}\, d\sigma \right]^*$$

$$\times\, e^{-j2\pi \Delta T u}\, e^{j2\pi \Delta f \tau}\, d\Delta T\, d\Delta f$$

$$\int d\sigma \int d\Delta\tau \int dt \;\; \tilde{f}\left(t+\tfrac{\Delta\tau}{2}\right)\tilde{f}^{*}\left(t-\tfrac{\Delta\tau}{2}\right)\tilde{f}^{*}\left(\sigma+\tfrac{\Delta\tau}{2}\right)\tilde{f}\left(\sigma-\tfrac{\Delta\tau}{2}\right) e^{-j2\pi\Delta\tau u} \underbrace{\int_{-\infty}^{\infty} e^{j2\pi\Delta f(\tau+\sigma-t)}\, d\Delta f}_{\delta(\tau+\sigma-t)}$$

$$\int d\sigma \int d\Delta\tau \left[\underbrace{\int_{-\infty}^{\infty} \tilde{f}\left(t+\tfrac{\Delta\tau}{2}\right)\tilde{f}^{*}\left(t-\tfrac{\Delta\tau}{2}\right)\delta(\tau+\sigma-t)\, dt}_{\tilde{f}\left(\tau+\sigma+\frac{\Delta\tau}{2}\right)\tilde{f}^{*}\left(\tau+\sigma-\frac{\Delta\tau}{2}\right)}\right] \tilde{f}^{*}\left(\sigma+\tfrac{\Delta\tau}{2}\right)\tilde{f}\left(\sigma-\tfrac{\Delta\tau}{2}\right) e^{-j2\pi\Delta\tau u}$$

$$= \iint \tilde{f}\left(\tau+\sigma+\tfrac{\Delta\tau}{2}\right)\tilde{f}^{*}\left(\sigma+\tfrac{\Delta\tau}{2}\right)\tilde{f}^{*}\left(\tau+\sigma-\tfrac{\Delta\tau}{2}\right)\tilde{f}\left(\sigma-\tfrac{\Delta\tau}{2}\right) d\sigma\; e^{-j2\pi\Delta\tau u}\, d\Delta\tau$$

call $\quad g(x) = \tilde{f}\left(\tau+x+\tfrac{\Delta\tau}{2}\right)\tilde{f}^{*}\left(x+\tfrac{\Delta\tau}{2}\right)$

$$= \int \Big[\underbrace{\int g(\sigma)\, g^{*}(\sigma-\Delta\tau)\, d\sigma}_{\text{convolution}}\Big] e^{-j2\pi\Delta\tau u}\, d\Delta\tau =$$

$$= \int \left[g(\Delta\tau) * g^{*}(-\Delta\tau)\right] e^{-j2\pi u\Delta\tau}\, d\Delta\tau = \text{F.T of convolution is multiplication in freq.}$$

$$= G(u)\cdot G^{*}(u) = |G(u)|^{2} =$$

$$= \left|\text{F.T.}\left[g(\sigma)\right]\right|^{2} = \left|\text{F.T.}\left[\tilde{f}\left(\tau+\sigma+\tfrac{\Delta\tau}{2}\right)\tilde{f}^{*}\left(\sigma+\tfrac{\Delta\tau}{2}\right)\right]\right|^{2}$$

$$= \left|\int_{-\infty}^{\infty} \tilde{f}\left(\tau+\sigma+\tfrac{\Delta\tau}{2}\right)\tilde{f}^{*}\left(\sigma+\tfrac{\Delta\tau}{2}\right) e^{-j2\pi u\sigma}\, d\sigma\right|^{2} = \text{change variables: } \sigma+\tfrac{\Delta\tau}{2}=\sigma'$$

$$= \left|\int_{-\infty}^{\infty} \tilde{f}(\tau+\sigma')\,\tilde{f}^{*}(\sigma')\, e^{-j2\pi u\sigma'}\, d\sigma' \cdot \underbrace{e^{-j2\pi u\Delta\tau/2}}_{1}\right|^{2}$$

$$= \left|\phi(\tau,u)\right|^{2} = \Theta(\tau,u)$$

It is worth noting that the converse statement is not true. The self-transform property does not guarantee that a particular function is an ambiguity function.

Problem 5

a) The transmitted signal is:

$$s(t) = w(t)\sin\left(a\ln(1-kt)\right) = 2\,\mathrm{Re}\left\{w(t)\left(\frac{-j}{2}\right)e^{ja\ln(1-kt)}\right\}$$

$$= 2\,\mathrm{Re}\left\{\frac{w(t)e^{-j\frac{\pi}{2}}}{2}e^{ja\ln(1-kt)}\right\} = 2\,\mathrm{Re}\left\{\frac{w(t)}{2}e^{j\left(-\frac{\pi}{2}+a\ln(1-kt)\right)}\right\}$$

$$= 2\,\mathrm{Re}\left\{\frac{w(t)}{2}e^{j\left(-\frac{\pi}{2}+a\ln(1-kt)-2\pi f_0 t\right)}e^{j2\pi f_0 t}\right\} \Rightarrow$$

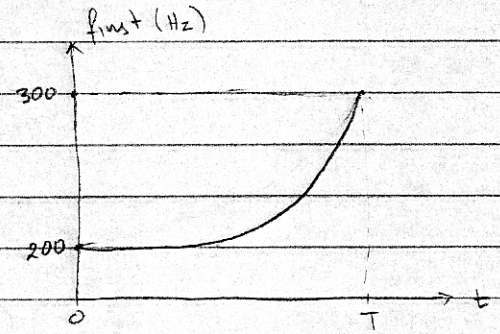
$$\tilde{s}(t) = \frac{w(t)}{2}e^{j\left(-\frac{\pi}{2}+a\ln(1-kt)-2\pi f_0 t\right)}, \quad 0 \le t \le T$$

so $\phi(t) = -\frac{\pi}{2} + a\ln(1-kt) - 2\pi f_0 t$

$$f_{inst} = \frac{1}{2\pi}\frac{d\phi(t)}{dt} = \frac{1}{2\pi}\left(\frac{-ka}{1-kt} - 2\pi f_0\right) =$$

$$= -f_0 - \frac{ka}{2\pi(1-kt)} \overset{a=-2\pi f_{min}/k}{=} -f_0 + \frac{2\pi f_{min}}{2\pi(1-kt)}$$

$$= -f_0 + \frac{f_{min}}{1-\frac{f_{max}-f_{min}}{f_{max}T}t} = -\frac{f_{min}+f_{max}}{2} + \frac{f_{min}}{1-\frac{f_{max}-f_{min}}{f_{max}T}t}$$

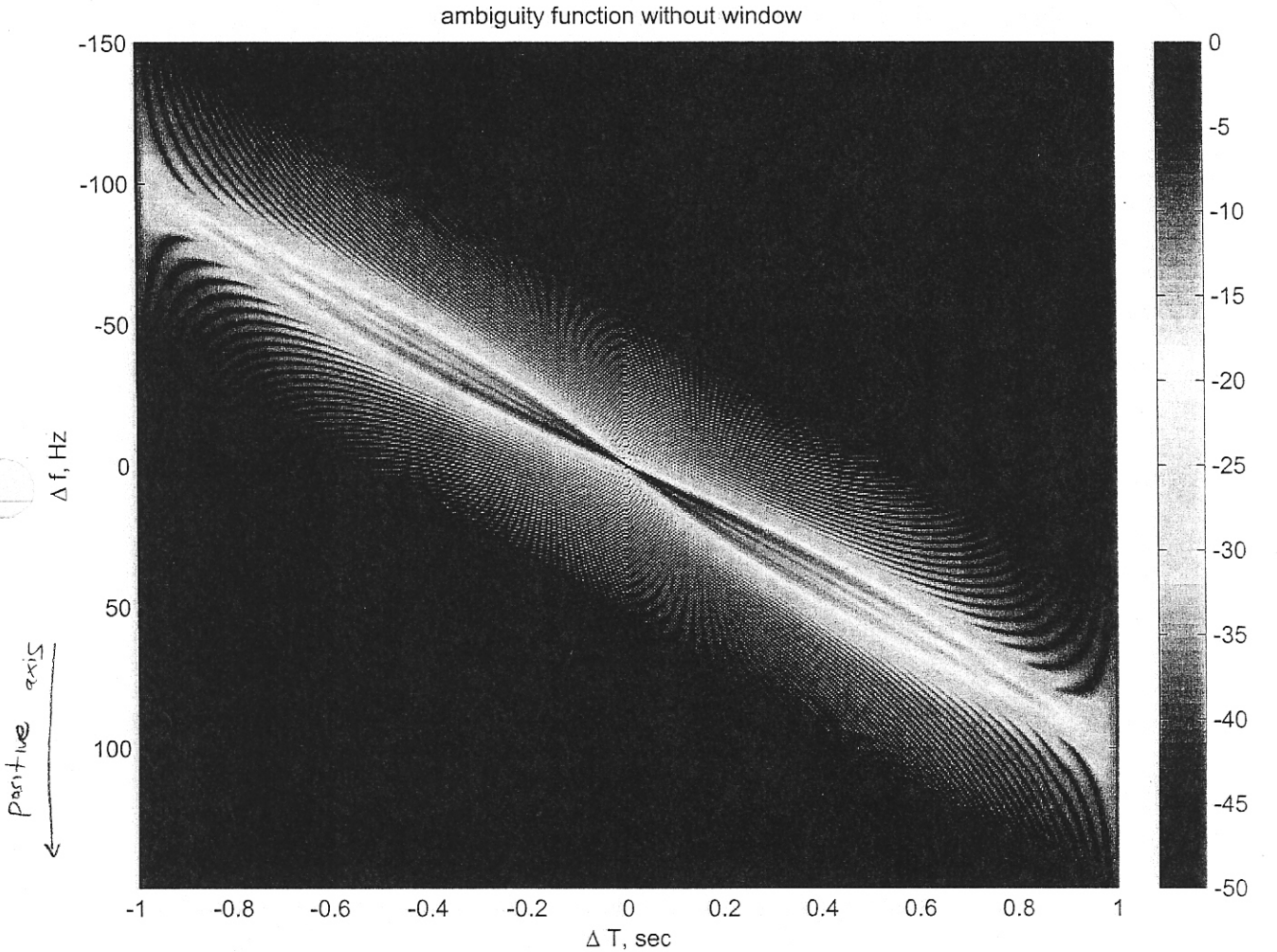


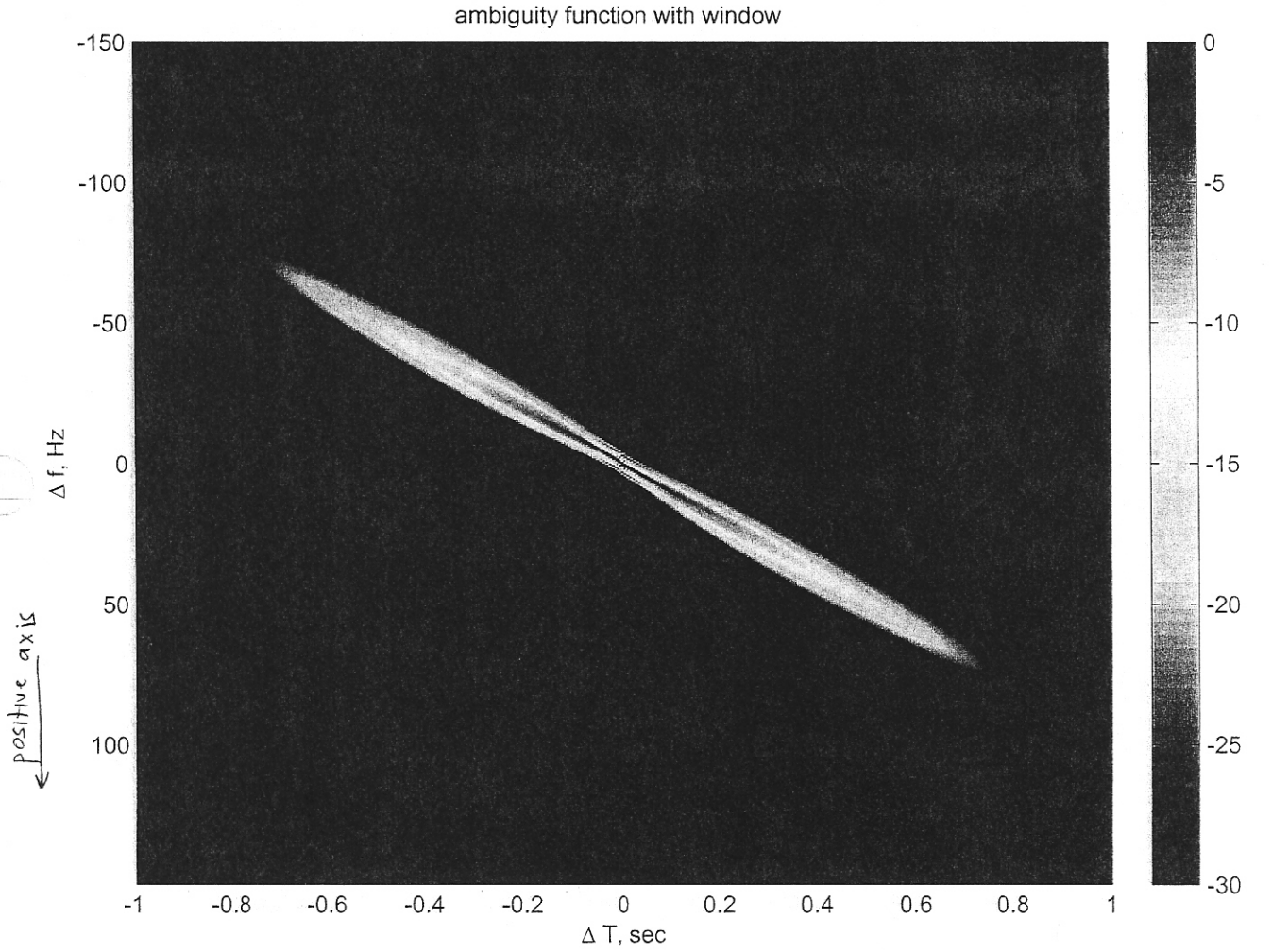
b) $$\phi(\Delta T, \Delta f) = \int \tilde{s}\left(t+\frac{\Delta T}{2}\right)\tilde{s}^*\left(t-\frac{\Delta T}{2}\right)e^{-j2\pi\Delta f t}\,dt$$

$$= \int w\left(t+\frac{\Delta T}{2}\right)\frac{1}{2}e^{j\left(-\pi/2-2\pi f_0\left(t+\frac{\Delta T}{2}\right)+a\ln\left(1-k\left(t+\Delta T/2\right)\right)\right)} \cdot \frac{1}{2}e^{-j\left(-\pi/2-2\pi f_0\left(t-\frac{\Delta T}{2}\right)+a\ln\left(1-k\left(t-\Delta T/2\right)\right)\right)} w\left(t-\frac{\Delta T}{2}\right) \cdot e^{-j2\pi\Delta f t}\,dt$$

$$= \frac{1}{4}e^{-j2\pi f_0\Delta T}\int w\left(t-\frac{\Delta T}{2}\right)w\left(t+\frac{\Delta T}{2}\right)e^{-ja\ln\left(\frac{1-k(t+\Delta T/2)}{1-k(t-\Delta T/2)}\right)}e^{-j2\pi\Delta f t}\,dt$$

```
function pltamb

max=300;    % Hz
fmin=200;
T=1;        % sec
Tw=.25*T;
k=(fmax-fmin)/fmax/T;
a=-2*pi*fmin/k;



% window function

t=linspace(0,T,fmax);
fs=fmax;                % sampling freq.
ind1=find(t<0.25);
ind2=find(t>0.75);


w_t(ind1)=sin(pi*t(ind1)/Tw/2).^2;
w_t(ind1(end)+1:ind2(1)-1)=1;
w_t(ind2)=sin(pi*(t(ind2)-T+2*Tw)/Tw/2).^2;

%windowed complex envelope

s_t=w_t.*exp(i*a*log(1-k*t));


% shift
N=2^nextpow2(length(t)+1);
for deltaT=0:length(t)-1

    s_t_shifted=[zeros(1,deltaT) s_t(1:end-deltaT)];
    p=s_t.*conj(s_t_shifted);

    % compute all Doppler shifts using fft
    T=fftshift(fft(p,N));
    Theta(:,deltaT+1)=abs(T).';

end
% normalize the ambiguity function to be 1 @ zero Doppler, zero range
Theta=Theta/max(max(Theta));

% Theta=flipud(Theta);
% fill out negative ranges from symmetry
Theta_total=[fliplr(flipud(Theta(:,2:end))) Theta];

imagesc([-t(end:-1:1) t(2:end)],fs*([0:N-1]/N-0.5),20*log10(abs(Theta_total)),[-30 0]) ;
colorbar
xlabel(texlabel('Delta T, sec')), ylabel(texlabel('Delta f, Hz')), title('ambiguity
function with window')
```


ambiguity function without window
-150
-100
-50
0
50
100
Δ f, Hz
positive axis
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
Δ T, sec
0
-5
-10
-15
-20
-25
-30
-35
-40
-45
-50


ambiguity function with window
-150
-100
-50
0
50
100
Δ f, Hz
positive axis
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
Δ T, sec
0
-5
-10
-15
-20
-25
-30

Problem 6

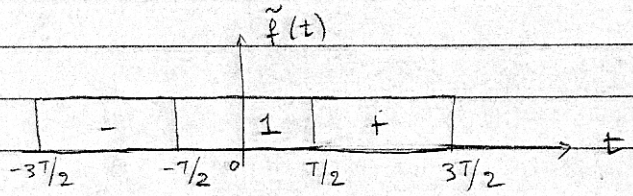
a) $$\phi(0,\Delta f) = \int_{-\infty}^{\infty} |\tilde{f}(t)|^2 e^{-j2\pi\Delta f t}\, dt$$

Note that $|\tilde{f}(t)| = \frac{1}{3T}$ for $-\frac{3T}{2} \le t \le \frac{3T}{2}$

thus $$\phi(0,\Delta f) = \int_{-3T/2}^{3T/2} \frac{1}{3T} e^{-j2\pi\Delta f t}\, dt = \text{sinc}(\pi 3T\Delta f)$$

$$\Rightarrow \Theta(0,\Delta f) = \text{sinc}^2(\pi 3T\Delta f)$$ Note that first zero @ $\Delta f = \pm 1/3T$

b)



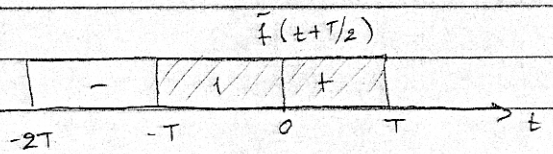
think that $\tilde{f}(t)$ has three blocks

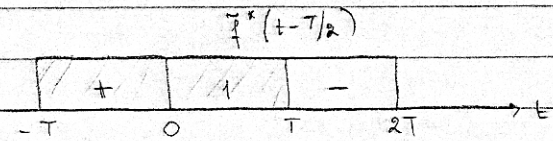
"−" denotes $\frac{1}{\sqrt{3T}} e^{-j2\pi\Delta F t}$

"1" denotes $\frac{1}{\sqrt{3T}}$

"+" denotes $\frac{1}{\sqrt{3T}} e^{j2\pi\Delta F t}$

$$\phi(T,0) = \int_{-\infty}^{\infty} \tilde{f}(t+\tfrac{T}{2})\, \tilde{f}^{*}(t-\tfrac{T}{2})\, dt$$





$$\phi(T,0) = \int_{-T}^{0} 1 \cdot \frac{e^{+j2\pi\Delta F(t+T/2)}}{3T}\, dt + \int_{0}^{T} 1 \cdot \frac{e^{+j2\pi\Delta F(t+\frac{T}{2})}}{3T}\, dt =$$

$$= \frac{1}{3T} \int_{0}^{T} e^{-j2\pi\Delta F(t+T/2)} + e^{j2\pi\Delta F(t+T/2)}\, dt$$

$$\Rightarrow \phi(T,0) = \frac{q}{3T}\int_0^T \cos\left(2\pi\Delta F\left(t+\frac{T}{2}\right)\right)dt = \frac{2}{3T}\,\frac{\sin(\pi\Delta F 3T)-\sin(\pi\Delta F T)}{2\pi\Delta F}$$

$$\Rightarrow \boxed{\Theta(T,0) = \left(\operatorname{sinc}(\pi\Delta F 3T) - \tfrac{1}{3}\operatorname{sinc}(\pi\Delta F T)\right)^2}$$

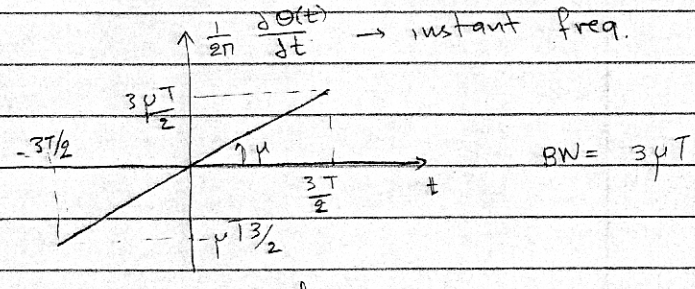
Similarly for $\phi(2T,0)$ only the "+" block overlaps from $-\frac{T}{2}$ to $\frac{T}{2}$ so

$$\phi(2T,0) = \int_{-T/2}^{T/2} \frac{1}{3T} e^{j2\pi\Delta F(t-T)}\cdot e^{j2\pi\Delta F(t+T)}\,dt = \frac{1}{3T}\int_{-T/2}^{T/2} e^{j4\pi\Delta F t}\,dt$$

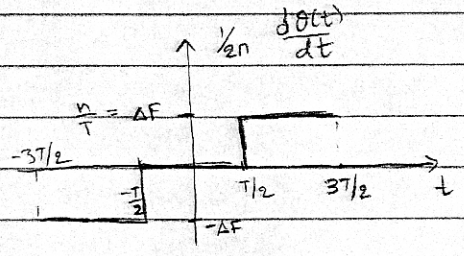
$$= \frac{1}{3}\operatorname{sinc}(2\pi\Delta F T) \Rightarrow \boxed{\Theta(2T,0) = \frac{1}{9}\operatorname{sinc}^2(2\pi\Delta F T)}$$

For $\phi(3T,0)$ there is no overlap so $\boxed{\Theta(3T,0)=0}$

c) Note that in continuous case we have a pulse which is multiplied by $e^{j2\pi\mu t^2/2}$ and we showed that the frequency is swept from $\frac{-3T}{2}$ to $\frac{3T}{2}$ as follows:



In our case we have a stepped sweep at two different frequencies : $\pm\frac{n}{T}$



$$\Theta(T,\Delta F) = \int_{-\infty}^{\infty} \tilde{f}\left(t-\frac{T}{2}\right)\tilde{f}^*\left(t+\frac{T}{2}\right)e^{-j2\pi\Delta F t}\,dt =$$

$$\Rightarrow \phi(T,\Delta F) = \int_{-T}^{0} 1\cdot \frac{e^{j2\pi\Delta F(t-T/2)}}{3T} e^{-j2\pi\Delta F t}\,dt + \int_{0}^{T} 1\cdot \frac{e^{j2\pi\Delta F(t+T/2)}}{3T} e^{-j2\pi\Delta F t}\,dt$$

$$= \int_{-T}^{0} \frac{e^{-j2\pi\Delta F T/2}}{3T}\,dt + \int_{0}^{T} \frac{e^{j2\pi\Delta F T/2}}{3T}\,dt = \frac{T}{3T}\left(e^{-j2\pi\Delta F T/2} + e^{j2\pi\Delta F T/2}\right)$$

$$= \frac{2}{3}\cos\left(2\pi\Delta F\frac{T}{2}\right) \Rightarrow \boxed{\Theta(T,\Delta F) = \frac{4}{9}\cos^2\left(2\pi\Delta F\frac{T}{2}\right)}$$

$$\phi(2T,2\Delta F) = \frac{1}{3T}\int_{-T/2}^{T/2} e^{j2\pi\Delta F(t-T)} e^{j2\pi\Delta F(t+T)} e^{-j2\pi 2\Delta F t}\,dt$$

$$= \frac{T}{3T} = \frac{1}{3} \Rightarrow \boxed{\Theta(2T,2\Delta F) = 1/9}$$

$$\phi(3T,3\Delta F) = 0 \Rightarrow \boxed{\Theta(3T,3\Delta F) = 0} \quad \text{since no overlap}$$

$$\phi(T,-\Delta F) = \int_{-T}^{0} \frac{1}{3T} e^{j2\pi\Delta F(t-T/2)} e^{j2\pi\Delta F t}\,dt + \int_{0}^{T} \frac{1}{3T} e^{j2\pi\Delta F(t+\frac{T}{2})} e^{j2\pi\Delta F t}\,dt$$

$$= \frac{2}{3T}\int_{0}^{T} \cos 2\pi 2\Delta F\left(t+\frac{T}{2}\right)dt = \frac{2}{3T}\,\frac{\sin\left(\pi 4\Delta F\frac{3T}{2}\right) - \sin\left(\pi 4\Delta F T/2\right)}{4\pi\Delta F}$$

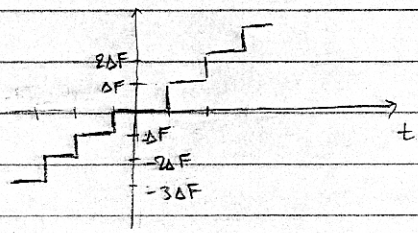
$$\Rightarrow \boxed{\Theta(T,-\Delta F) = \left[\operatorname{sinc}\left(\pi 4\Delta F\frac{3T}{2}\right) - \frac{1}{3}\sin\left(4\pi\Delta F T/2\right)\right]^2}$$

$$\phi(2T,-2\Delta F) = \frac{1}{3T}\int_{-T/2}^{T/2} e^{j2\pi 2\Delta F t} e^{j2\pi 2\Delta F t}\,dt = \frac{1}{3}\operatorname{sinc}(4\pi\Delta F\cdot T)$$

$$\Rightarrow \boxed{\Theta(2T,-2\Delta F) = \frac{1}{9}\operatorname{sinc}^2(4\pi\Delta F T)}$$

for $m \geq 3$ $\quad \boxed{\Theta(3T,-3\Delta F) = 0}$

d) The generalization from 3 digit to N digit has intant freq.



It tries to approximate the LFM, thus we get the advantages of the LFM

In part (a) we showed that

$$\Theta(0, \Delta f) = \mathrm{sinc}^2(\pi 3T \Delta f)$$

thus the Doppler accuracy $\sim \frac{1}{3T}$. Generalizing to N digits the Doppler accuracy will be $\sim \frac{1}{NT}$.

$$\phi(\Delta T, 0) = \int_{-\infty}^{\infty} \tilde{f}(t)\, \tilde{f}^*(t-\Delta T)\, dt \overset{0<\Delta T<3T}{=}$$

$$= \frac{1}{3T}\left[\int_{-\frac{3T}{2}+\Delta T}^{-T/2} e^{-j2\pi\Delta F \Delta T}\, dt + \int_{-T/2}^{-T/2+\Delta T} e^{j2\pi\Delta F(t-\Delta T)}\, dt + \int_{-T/2+\Delta T}^{T/2} 1\, dt + \int_{T/2}^{T/2+\Delta T} e^{j2\pi\Delta F t}\, dt \right.$$

$$\left. + \int_{T/2+\Delta T}^{3T/2} e^{j2\pi\Delta F \Delta T}\, dt\right] = \frac{1}{3T}\left[(T-\Delta T)\left(1+2\cos(2\pi\Delta F \Delta T)\right) - 2\Delta T \,\mathrm{sinc}(\pi\Delta F\Delta T)\left(\sin\pi\Delta F\Delta T - \cos(\pi\Delta F\Delta T)\cos(\pi\Delta F T)\right)\right]$$

Note that for $\Delta T = \frac{n}{\Delta F}$, $\phi\left(\frac{n}{\Delta F}, 0\right) = \frac{T - n/\Delta F}{T} = 0$

Generalizing to N digits the range resolution is $\sim \frac{n}{N\Delta F} = \frac{n}{\mu NT}$

Whether Doppler is known or unknown increasing N is always a good strategy because both Doppler and range are inversely proportional to N. Better results are obtained when Doppler is known rather than unknown because of the pulse compression property (range accuracy $\sim \frac{1}{N\mu T}$)