Quiz # 2 - solutions

Problem 1

a) Consider the output of the integrator :

$$\hat{h}(T_0) = \int_{-T/2}^{T/2} \frac{y(t)\,w(t-T_0)}{T}\,dt = \frac{1}{T}\int_{-T/2}^{T/2} dt \int_{-\infty}^{\infty} (x(t-\tau)+w(t-\tau))h(\tau)d\tau\; w(t-T_0)$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} dt \int_{-\infty}^{\infty} x(t-\tau)w(t-T_0)\,h(\tau)\,d\tau + \frac{1}{T}\int_{-T/2}^{T/2} dt \int_{-\infty}^{\infty} w(t-\tau)\,h(\tau)\,w(t-T_0)\,d\tau$$

$$= \int_{-\infty}^{\infty} d\tau\, h(\tau) \int_{-T/2}^{T/2} \frac{x(t-\tau)\,w(t-T_0)}{T}\,dt + \int_{-\infty}^{\infty} h(\tau)\,d\tau \int_{-T/2}^{T/2} \frac{w(t-\tau)\,w(t-T_0)}{T}\,dt$$

as $T \to \infty$ then

$$\hat{h}(T_0) = \int_{-\infty}^{\infty} d\tau\, h(\tau)\; E[x(t-\tau)w(t-T_0)] + \int_{-\infty}^{\infty} h(\tau)\,d\tau\; E[w(t-\tau)w(t-T_0)]$$

since $x(t)$, $w(t)$ are 1st and 2nd order ergodic processes. Also $x(t)$, $w(t)$ are zero mean and uncorrelated so $E[x(t)w(t)] = E[x(t)]\,E[w(t)] = 0$ thus

$$\hat{h}(T_0) = \int_{-\infty}^{\infty} h(\tau)\,\delta(T_0-\tau)\,d\tau = h(T_0) \qquad (\text{I assume } \sigma_w^2 = 1).$$

Hence we have the desired result.

Note that $E[\hat{h}(T_0)] = h(T_0)$ otherwise ergodicity would not be a valid assumption.

b) $\hat{h}(T_0) = \frac{1}{T}\int_{-T/2}^{T/2} y(t)\,w(t-T_0)\,dt$ is the estimate of the impulse response $\forall\, T_0 \in \mathbb{R}$. Following the spirit of B-T method the estimate of the transfer function $H(f)$ is:

$$\hat{H}(f) = \int_{-\infty}^{\infty} z(T_0)\,\hat{h}(T_0)\,e^{-j2\pi f T_0}\,dT_0.$$

where $z(t)$ is the window function.

$$E[\hat{H}(f)] = \int_{-\infty}^{\infty} z(T_0)\,E[\hat{h}(T_0)]\,e^{-j2\pi f T_0}\,dT_0 = \int_{-\infty}^{\infty} z(T_0)\,h(T_0)\,e^{-j2\pi f T_0}\,dT_0$$

$$= \int_{-\infty}^{\infty} Z(u)\,H(f-u)\,du \simeq H(f)$$

under the assumption that $H(f)$ is smooth (magnitude and phase) across the window bandwidth.

Note that $y(t) = (x(t) + w(t)) * h(t) = x(t) * h(t) + w(t) * h(t)$ Since $x(t)$ is Gaussian random process then $x(t) * h(t)$ is a Gaussian random process because Gaussianity is preserved under linear transformations. Similarly $w(t) * h(t)$ is a Gaussian random process. Also the processes $x(t) * h(t)$ and $w(t) * h(t)$ are independent because $x(t)$, $w(t)$ are independent. Hence $y(t)$ is a Gaussian random process with $E[y(t)] = 0$ and

$$\sigma_y^2 = E[y^2(t)] = \int_{-\infty}^{\infty} |H(f)|^2 \left(S_x(f) + 1\right) df$$

$$\sigma^2_{\hat{H}(f)} = E\left[|\hat{H}(f)|^2\right] - \left|E\left[\hat{H}(f)\right]\right|^2$$

$$E\left[|\hat{H}(f)|^2\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} z(T_0)\, z(T_0')\, E\left[\hat{h}(T_0)\hat{h}(T_0')\right] e^{-j2\pi f(T_0-T_0')}\, dT_0\, dT_0'$$

$$\left|E\left[\hat{H}(f)\right]\right|^2 = E\left[\hat{H}(f)\right] E\left[\hat{H}^*(f)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} z(T_0)\, z(T_0')\, \underbrace{E\left[\hat{h}(T_0)\right]}_{h(T_0)}\, \underbrace{E\left[\hat{h}(T_0')\right]}_{h(T_0')}\, e^{-j2\pi f(T_0-T_0')}\, dT_0\, dT_0'$$

so $$\sigma^2_{\hat{H}(f)} = \iint z(T_0)\, z(T_0') \left[E\left[\hat{h}(T_0)\hat{h}(T_0')\right] - h(T_0)h(T_0')\right] e^{-j2\pi f(T_0-T_0')}\, dT_0\, dT_0'$$

and $$E\left[\hat{h}(T_0)\hat{h}(T_0')\right] = \frac{1}{T^2}\int_{-T/2}^{T/2} dt_1 \int_{-T/2}^{T/2} dt_2\; \underbrace{E\left[y(t_1)w(t_1-T_0)\,y(t_2)\,w(t_2-T_0')\right]}_{\text{invoking moment thm.}}$$

$$E\left[y(t_1)w(t_1-T_0)\right]E\left[y(t_2)w(t_2-T_0')\right] + E\left[y(t_1)y(t_2)\right]E\left[w(t_1-T_0)w(t_2-T_0')\right]$$
$$+\; E\left[y(t_1)w(t_2-T_0')\right]E\left[y(t_2)w(t_1-T_0)\right] =$$
$$= h(T_0)\,h(T_0') + R_y(t_1-t_2)\,\delta\left(t_1-t_2-(T_0-T_0')\right) + h(t_1-t_2+T_0')\,h(t_2-t_1+T_0)$$

$$E\left[\hat{h}(T_0)\hat{h}(T_0')\right] = \boxed{\frac{1}{T^2}\int_{-T/2}^{T/2} dt_1 \int_{-T/2}^{T/2} dt_2\; h(T_0)\,h(T_0')} \Longrightarrow h(T_0)\,h(T_0')$$

so this is cancelled out by the mean

$$+\; \boxed{\frac{1}{T^2}\int_{-T/2}^{T/2} dt_1 \int_{-T/2}^{T/2} dt_2\; R_y(t_1-t_2)\,\delta\left(t_1-t_2-(T_0-T_0')\right)}$$

change variables $t_1-t_2=\tau$, $t_2=\tau'$ $$\Longrightarrow \frac{1}{T^2}\int_{-T}^{T} d\tau \int_{-\frac{(T-|\tau|)}{2}}^{\frac{T-|\tau|}{2}} d\tau'\; R_y(\tau)\,\delta\left(\tau-(T_0-T_0')\right) = \underbrace{\left(\frac{T-|T_0-T_0'|}{T^2}\right)R_y(T_0-T_0')}_{=R_y'(T_0-T_0')}$$

$$+\; \boxed{\frac{1}{T^2}\int_{-T/2}^{T/2} dt_1 \int_{-T/2}^{T/2} dt_2\; h(t_1-t_2+T_0')\,h(t_2-t_1+T_0)}$$

change variables $t_1-t_2=\tau$, $t_2=\tau'$ $$\Longrightarrow \frac{1}{T^2}\int_{-T}^{T} d\tau \int_{-\frac{T-|\tau|}{2}}^{\frac{T-|\tau|}{2}} h(\tau+T_0')\,h(-\tau+T_0)\,d\tau'$$

$$= \int_{-T}^{T}\left(\frac{T-|\tau|}{T^2}\right)h(\tau+T_0')\,h(-\tau+T_0)\,d\tau$$ $\longrightarrow$ something like a convolution. In Fourier domain will be $\frac{1}{T}\,|H(f)|^2 \sin^2\left(2\pi f\frac{T}{2}\right)$

Thus:

$$\sigma^2_{\hat{H}(f)} = \iint z(T_0)\, z(T_0') \left[\underbrace{R_y'(T_0 - T_0')}_{\Downarrow} + \int_{-T}^{T} \left(\frac{T-|\tau|}{T^2}\right) h(\tau+T_0)\, h(-\tau+T_0)\, d\tau \right] e^{-j2\pi f (T_0 - T_0')}\, dT_0\, dT_0'$$

$$\left(\frac{T - |T_0 - T_0'|}{T^2}\right) \int_{-\infty}^{\infty} \left(S_x(u) + 1\right) |H(u)|^2 e^{j2\pi u (T_0 - T_0')}\, du$$

this term will lead to $\longrightarrow$ $\left(S_x(f)+1\right)|H(f)|^2 \int_{-T/2}^{T/2} |z_1(t)|^2\, dt$

(I have to assume smoothness of $S_x(f)$ also)

this term will lead to $\longrightarrow$ $|H(f)|^2 \int_{-T/2}^{T/2} |z_2(t)|^2\, dt$

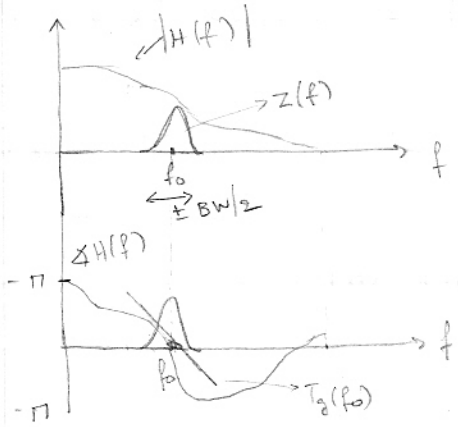
where $z_1(t)$, $z_2(t)$ some composite tapers related to $z(t)$

So eventually I will have something like:

$$\sigma^2_{\hat{H}(f)} = |H(f)|^2 \left[\left(S_x(f)+1\right) \frac{C_{w_1}}{T} + \frac{C_{w_2}}{T} \right]$$

c) We saw that $E\left[\hat{H}(f)\right] = \int H(u)\, Z(f-u)\, du \simeq H(f)$

In order to have unbiased estimate, $H(f)$ must have smooth magnitude and phase across the window bandwidth.



If we pick a frequency f_0 we have:

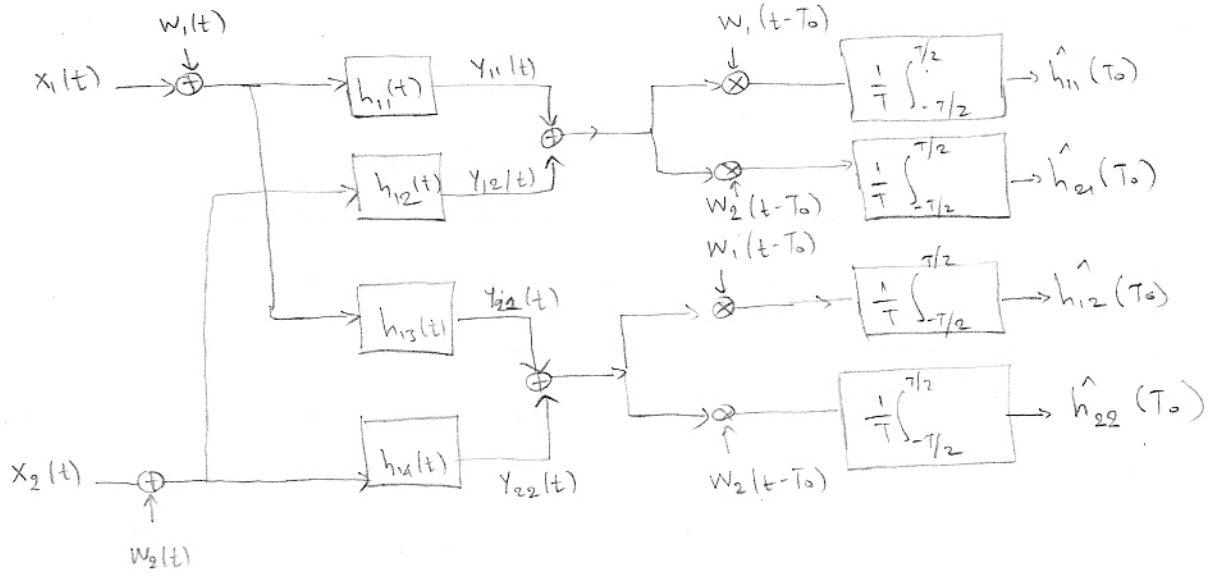
$$E\left[\hat{H}(f_0)\right] = \int_{f_0 - BW/2}^{f_0 + BW/2} H(u)\, Z(f_0 - u)\, du$$

$$\approx \int_{-BW/2}^{BW/2} |H(f_0)| e^{j \measuredangle H(f_0)} e^{-j 2\pi f T_g(f_0)} Z(f)\, df$$

$$= |H(f_0)| e^{j \measuredangle H(f_0)} \int_{-BW/2}^{BW/2} e^{-j 2\pi f T_g(f_0)} Z(f)\, df$$

we require $2\pi \frac{BW}{2} T_g(f_0) \leq \frac{\pi}{4}$ (same quadrant) $\Rightarrow BW \leq \frac{1}{4 T_g(f_0)}$ or more stringent $\boxed{BW \leq \frac{1}{4\left[\max_f T_g(f)\right]}, \forall f}$ in order for the phase of $H(f)$ not to change significantly across the bandwidth of the window.

d) Consider the following structure:



Note that

$$E\left[\hat{h}_{11}(T_0)\right] = E\left[\int_{-T/2}^{T/2} \left(y_{11}(t) + y_{21}(t)\right) w_1(t-T_0)\, dt\right] = E\int_{-T/2}^{T/2} dt \int_{-\infty}^{\infty} h_{11}(\tau)\left(x_1(t-\tau) + w_1(t-\tau)\right) d\tau\, w_1(t-T_0) +$$

uncorrelated

$$\int_{-T/2}^{T/2} dt \int_{-\infty}^{\infty} h_{21}(\tau)\left(x_2(t-\tau) + w_2(t-\tau)\right) d\tau\, w_1(t-T_0) = \int_{-\infty}^{\infty} h_{11}(\tau)\, \delta(T_0-\tau)\, d\tau = h_{11}(T_0)$$

0 uncorrelated

Similarly, $E\left[\hat{h}_{12}(T_0)\right] = h_{12}(T_0)$, $E\left[\hat{h}_{21}(T_0)\right] = \hat{h}_{21}(T_0)$, $E\left[\hat{h}_{22}(T_0)\right] = h_{22}(T_0)$.

The variances can be computed as in part (b).

Problem 2

In this problem we compute spectral estimates based on data provided. The approach chosen in these solutions was to implement the segmented frequency averaging. We will see that breaking the data into segments helps reduce the variance as does frequency averaging. There is a trade off however. If the segments are too small, the assumption of smoothness of the spectra is violated leading to large biases.
A Hanning window has been chosen for use because it has desirable sidelobe properties. The max sidelobe level for a Hanning window is -32 dB (see Harris' paper) as opposed to -13 dB for a rectangular window. This will help to prevent "leakage" if the data contains closely spaced tones.
One point that should be mentioned: no overlapping of the data snapshots was used. In practice, when windowing is used data segments are overlapped. It was not used here in order to keep the statistical results more tractable.

Statistics of Estimates

We want to compute the bias and the variance of $\hat{S}_x(f)$. The segment length is T_L. Freq. averaging is done across $2N+1$ points.

$$\hat{S}_x(f) = \frac{1}{L} \sum_{i=1}^{L} \sum_{n=-N}^{N} h_n \left| X_w^i\left(f + \frac{n}{T_L}\right)\right|^2$$

where $$X_w^i(f) = \int_{T_i - T_L/2}^{T_i + T_L/2} w(t - T_i) \cdot x(t)\, e^{-j2\pi f t}\, dt$$

where $w(t)$ is the Hanning window of duration T_L centered at T_i for each segment.

$$E\left[\hat{S}_x(f)\right] = \frac{1}{L} \sum_{i=1}^{L} \sum_{n=-N}^{N} h_n \, E\left[\left|X_W^i\left(f + \frac{n}{T_L}\right)\right|^2\right]$$

$$= \frac{1}{L} \sum_i \sum_n h_n \int_{T_i - \frac{T_L}{2}}^{T_i + \frac{T_L}{2}} dt_1 \int_{T_i - T_L/2}^{T_i + \frac{T_L}{2}} dt_2 \; \underbrace{E\left[x(t_1)x(t_2)\right]}_{R_x(t_1 - t_2)} w(t_1 - T_i)\, w^*(t_2 - T_i)\, e^{-j2\pi\left(f + \frac{n}{T_L}\right)(t_1 - t_2)}$$

$R_x(t_1 - t_2) \rightarrow$ assuming WSS process

$$= \int_{-\infty}^{+\infty} S_x(u)\, e^{j2\pi u(t_1 - t_2)}\, du$$

invoking smoothness over freq. band of averaging we have.

$$E\left[\hat{S}_x(f)\right] = \frac{1}{L} \sum_i \sum_n h_n \, S_x(f) \int_{-\infty}^{\infty} W\left(f + \frac{n}{T_L} - u\right) W^*\left(f + \frac{n}{T_L} - u\right) du$$

observe that the phase shifts due to window delays are cancelled out.

$$= \frac{1}{L} \sum_i \sum_n h_n \, S_x(f) \int_{-T_L/2}^{T_L/2} |w(t)|^2 \, dt$$

$$= S_x(f) \sum_{n=-N}^{N} h_n \frac{C_W}{T_L} \qquad (C_W = 1.5 \text{ from Harris paper})$$

So for unbiased estimate assuming constant $h_n = h$ we have $(2N+1)\, h \frac{1.5}{T_L} = 1 \Rightarrow h = \frac{T_L}{1.5\,(2N+1)}$

Note that we assumed $S_x(f)$ smooth over $(2N+1)\frac{2}{T_L}$

To compute the variance of $S_x(f)$ we use the key result for Gaussian real processes :

$\sigma^2_{|X_W|^2} = \left(\sigma^2_{X_W}\right)^2$ conditioned we are away from DC i.e. $\left|f + \frac{N}{T_L}\right| > B_W/2$

(Close to DC the variance is higher)

Similar to lecture notes we have:

$$\sigma^2_{\hat{S}_x(f)} = \frac{1}{L^2} \sum_{i=1}^{L} \sum_{j=1}^{L} \sum_{n_1=-N}^{N} \sum_{n_2=-N}^{N} h^i_{n_1} h^j_{n_2} \left| S_x(f) \int_{-\infty}^{\infty} W(u) W^*\left(u - \frac{n_1-n_2}{T}\right) e^{-j2\pi(T_i-T_j)u} du \right|^2$$

define
$$\rho^{ij}_{n_1 n_2} = \frac{\int_{-\infty}^{\infty} W(u) W\left(u - \frac{n_1-n_2}{T_L}\right) e^{-j2\pi u(T_i-T_j)} du}{\int_{-\infty}^{\infty} |W(u)|^2 du}$$

Note that $|\rho^{ij}_{n_1 n_2}| < 1$.

$\sigma^2_{\hat{S}_x(f)} / S^2(f)$ can be written in matrix form as follows:

$$\frac{\sigma^2_{\hat{S}(f)}}{S^2(f)} = \frac{1}{L^2} \underbrace{\begin{bmatrix} \underline{h}' \\ \underline{h}' \\ \vdots \\ \underline{h}' \end{bmatrix}^T}_{1 \times L(2N+1)} \underbrace{\begin{bmatrix} \left[|\rho^{11}_{n_1 n_2}|^2\right] & \overbrace{\left[|\rho^{12}_{n_1 n_2}|^2\right]}^{2N+1 \times 2N+1} & \cdots \\ \vdots & \left[|\rho^{ii}_{n_1 n_2}|^2\right] & \\ & & \ddots \end{bmatrix}}_{(2N+1)L \times (2N+1)L} \begin{bmatrix} \underline{h}' \\ \underline{h}' \\ \vdots \\ \underline{h}' \end{bmatrix} \quad (H^T K H)$$

where $\underline{h}'$ is a vector of length $2N+1$ whose elements are $\frac{h_n}{\sum_n h_n}$

If $h_n = h = \frac{T_L}{C_W(2N+1)}$ then $\sum h_n = \frac{T_L}{C_W}$ and $h'_n = \frac{1}{2N+1}$

$$\rho^{ii}_{n_1 n_2} = \frac{\int_{-T_L/2}^{T_L/2} W(u) W^*\left(u - \frac{n_1-n_2}{T_L}\right) du}{\int_{-\infty}^{\infty} |W(u)|^2 du} = \frac{T}{C_W} \int_{-T_L/2}^{T_L/2} |w(t)|^2 e^{-j2\pi\frac{(n_1-n_2)}{T_L}} dt$$

For the Hanning window:

$$w(t) = \frac{2}{T_L} \cos^2\left(\frac{\pi t}{T_L}\right) \qquad |t| \le T_L/2$$

$$\bar{W}(f) = \frac{2}{T} F\left\{ \frac{1}{2} + \frac{1}{2}\cos\frac{2\pi t}{T_L} \right\} = \frac{2}{T}\left[\frac{\delta(f)}{2} + \frac{1}{4}\delta\left(f - \frac{2\pi}{T_L}\right) + \delta\left(f + \frac{2\pi}{T_L}\right)\right]$$

$$* \quad T_L \operatorname{sinc}\left(2\pi f \frac{T_L}{2}\right)$$

$$\Rightarrow \bar{W}(f) = \operatorname{sinc}\left(\frac{2\pi f T_L}{2}\right) + \frac{1}{2}\operatorname{sinc}\left(2\pi f \frac{T_L}{2} - \pi\right) + \frac{1}{2}\operatorname{sinc}\left(2\pi f \frac{T_L}{2} + \pi\right)$$

Note that the width of the main lobe is $\pm \frac{2}{T_L}$ so if $\frac{n_1 - n_2}{T_L} > \frac{2}{T_L}$ then the correlation goes to 0.

if $n_1 = n_2 \rightarrow \dfrac{\int_{-\infty}^{\infty} W(u) W^*(u-0)\, du}{C_W/T} = 1$

if $|n_1 - n_2| = 1 \rightarrow \dfrac{\int_{-\infty}^{\infty} \bar{W}(u) \bar{W}^*\left(u - \frac{1}{T_L}\right) du}{C_W/T} = 2/3$

if $|n_1 - n_2| = 2 \rightarrow \rho^{ii}_{n_1 n_2} = 1/6$

if $|n_1 - n_2| > 2 \rightarrow \rho^{ii}_{n_1 n_2} = 0$

Note that for frequency averaging the min spacing is $\frac{4}{T_L}$.
It is easy to compute the matrix $[|\rho^{ii}_{n_1 n_2}|^2]$

$$\left[|\rho^{ii}_{n_1 n_2}|^2 \right] = \begin{bmatrix} 1 & (2/3)^2 & (1/6)^2 & \cdots & 0 \\ (2/3)^2 & \ddots & \ddots & \ddots & (1/6)^2 \\ (1/6)^2 & \ddots & \ddots & \ddots & (2/3)^2 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & & (1/6)^2 & (2/3)^2 & 1 \end{bmatrix}_{2N+1 \times 2N+1}$$

and $\frac{1}{L^2} \underline{h}'^T \left[|\rho^{ii}_{n_1 n_2}|^2 \right] \underline{h}' = \dfrac{1}{L^2 (2N+1)^2}\left[(2N+1)(1)^2 + (2N)\, 2\left(\frac{2}{3}\right)^2 + (2N-1)\, 2\left(\frac{1}{6}\right)^2 \right]$

It easy to conclude that $\sigma^2_{\hat{S}(f)} \sim \dfrac{S^2(f)}{N L^2}$.

$S(f)$ is a real function so no consideration must be given to the phase of the estimate. If the spectrum appears

to have narrowband phenomena then we need to adjust the segment length is order not to violate the smoothness assumption. In our case, there exist two tonals at 5 Hz and 10 Hz and a narrowband component at 40 Hz.

Figure 1 shows the estimate using no averaging ($N=0$) in order to see the tonals clearly.

Figure 2 shows the estimate without averaging but increased segments ($T_{seg} = 40$ sec). We note that the variance is increased relative to figure 1 because we do less segment averaging.

Figure 3 shows the estimate with $N=3$ but reduced segments ($T_{seg} = 5$ sec). The variance is reduced at the expense of resolution.

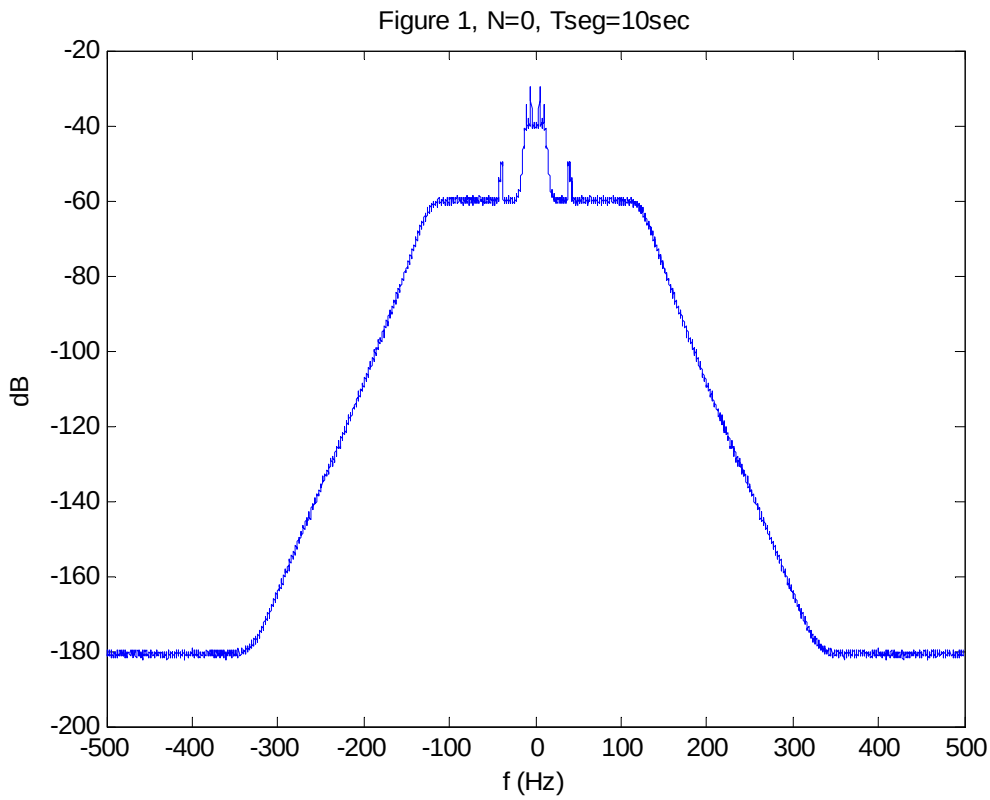


Figure 1, N=0, Tseg=10sec
-20
-40
-60
-80
-100
-120
-140
-160
-180
-200
dB
-500
-400
-300
-200
-100
0
100
200
300
400
500
f (Hz)

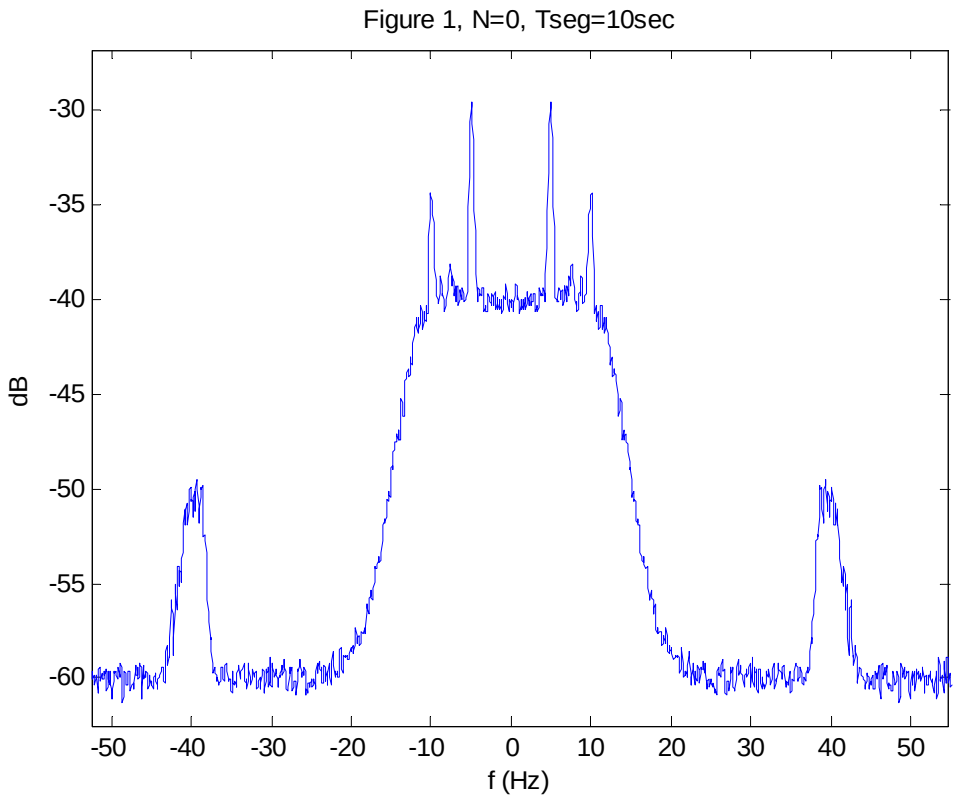


Figure 1, N=0, Tseg=10sec
-30
-35
-40
-45
-50
-55
-60
dB
-50
-40
-30
-20
-10
0
10
20
30
40
50
f (Hz)

Figure 2, N=0, Tseg=40sec

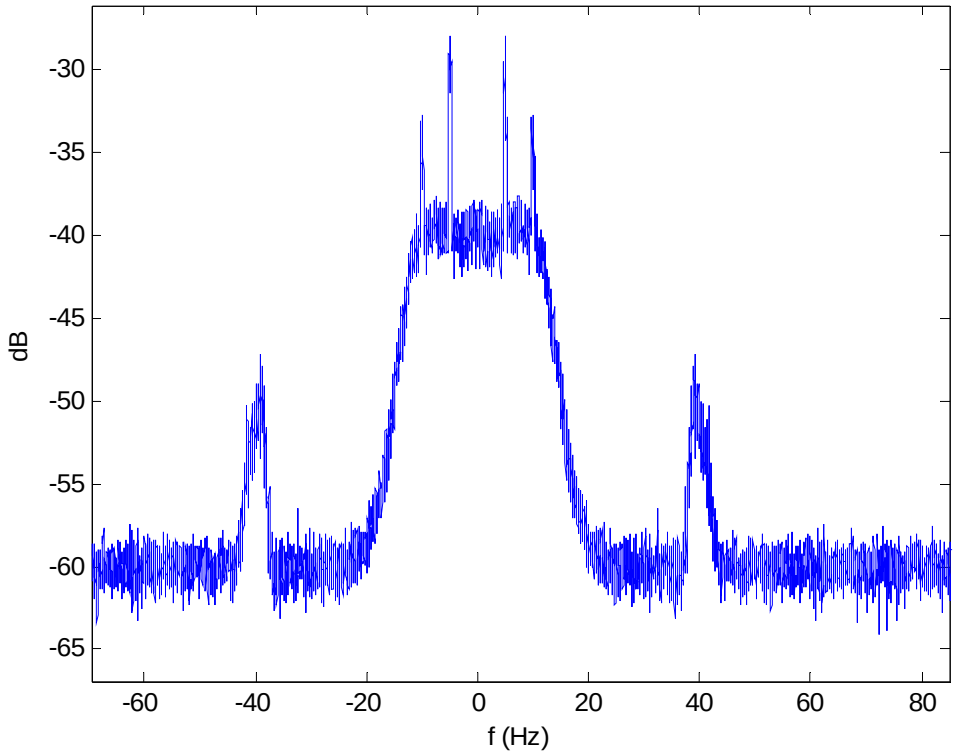
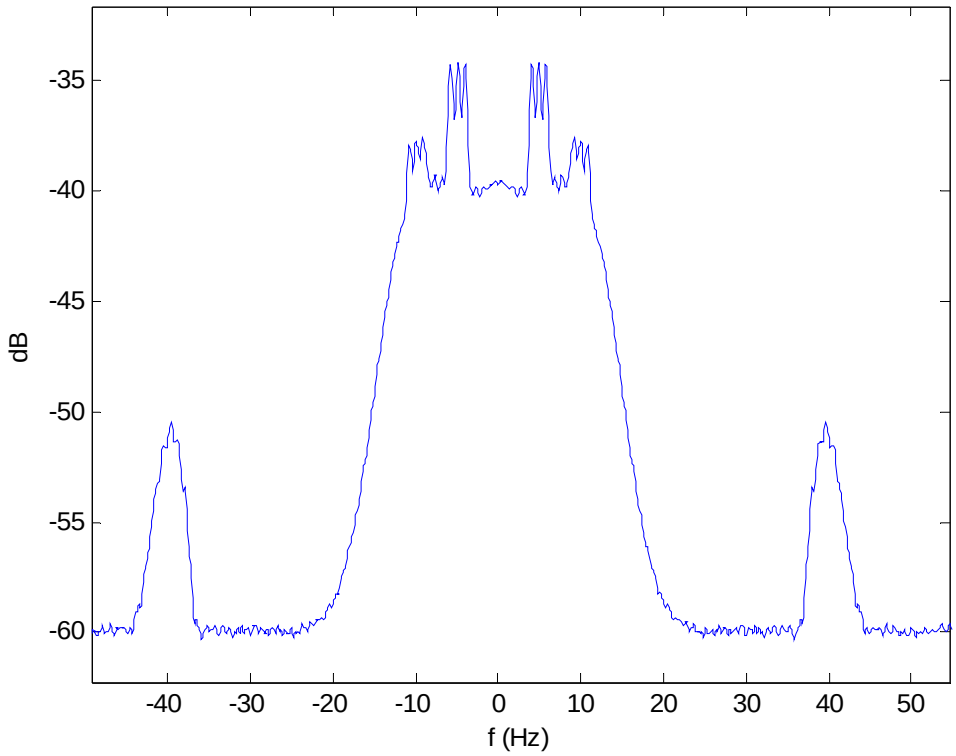


Figure 3, N=1, Tseg=5sec



```
function psd_fft

load data;
% column vector of data
fs=1000;
% sampling frequency of data in Hz
T=1000;
% duration of data in sec -> foundamental freq. limitation to 1/T Hz.
N=1;
% averaging in freq. (total 2N+1  points)
Tseg=5;
% segment of data in sec

w=(2/Tseg)*cos(pi*linspace(-Tseg/2,Tseg/2,Tseg*fs)'/Tseg).^2;
% use Hanning window
Cw=1.5/Tseg;
% from Harris paper
h=ones(1,2*N+1)/Cw/(2*N+1);
% condition for unbiased estimate

Nfft=2^nextpow2(fs*Tseg);
% fft points


df=fs/Nfft;
% freq. resolution for plotting
res=4/Tseg;
% mainlobe width for Hanning
M=ceil(res/df);
% min spacing to average uncorrelated r.v.

S=zeros(Nfft,1);
K=floor(T/Tseg);
% loop number
for i=1:K

seg=data(1+(i-1)*fs*Tseg:i*fs*Tseg);
% take one segment of data at a time

 Xw=fftshift(fft(w.*seg,Nfft));
 Y=conj(Xw).*Xw;


Y=[zeros(N*M,1); Y; zeros(N*M,1)];
% append zeros to acount for averaging at the limits

    for k=1:Nfft
        S(k)=S(k) + h*Y(k:M:2*N*M+k)/K;
    end

end


plot([-.5:1/Nfft:.5-1/Nfft]*fs,10*log10(S/fs^2));
% normalization for estimate
title('Figure 3, N=1, Tseg=5sec')  ,xlabel('f (Hz)'), ylabel('dB')
```