Problem 1

(a) Consider the output of the integrator:

\[ h(T_o) = \int_{-T/2}^{T/2} y(t) w(t-T_o) \, dt = \int_{-T/2}^{T/2} \left( x(t) + w(t-T_o) \right) \, dt \]

\[ = \frac{1}{T} \int_{-T/2}^{T/2} x(t) \, w(t-T_o) \, dt + \frac{1}{T} \int_{-T/2}^{T/2} w(t-T_o) \, dt \]

\[ = \int_{-\infty}^{\infty} h(z) \, \left( \frac{x(t-z) \, w(t-T_o)}{T} \right) \, dt + \int_{-\infty}^{\infty} h(z) \, \left( \frac{w(t-z) \, w(t-T_o)}{T} \right) \, dt \]

as \( T \to \infty \) then

\[ h(T_o) = \int_{-\infty}^{\infty} h(z) \, E \left[ x(t-z) \, w(t-T_o) \right] + \int_{-\infty}^{\infty} h(z) \, \delta(t-T_o) \, dt = h(T_o) \]

since \( x(t), w(t) \) are 1st and 2nd order ergodic processes. Also \( x(t), w(t) \) are zero mean and uncorrelated so

\[ E \left[ x(t) \, w(t) \right] = E(x(t)) \, E(w(t)) = 0 \]

thus

\[ h(T_o) = \int_{-\infty}^{\infty} h(z) \, \delta(t-z) \, dt = h(T_o) \]

(I assume \( \sigma_w^2 = 1 \)).

Hence, we have the desired result.

Note that \( E[h(T_o)] = h(T_o) \) otherwise ergodicity would not be a valid assumption.
b) $\hat{h}(T_0) = \frac{i}{\pi} \int_{-T/2}^{T/2} y(t) \hat{w}(t-T_0) dt$ is the estimate of the impulse response $h(t)$ at $T_0$. Following the spirit of B-T method, the estimate of the transfer function $H(f)$ is:

$$\hat{H}(f) = \int_{-\infty}^{\infty} z(T_0) \hat{h}(T_0) e^{-j2\pi f T_0} dT_0,$$

where $z(t)$ is the window function.

$$E[\hat{H}(f)] = \int_{-\infty}^{\infty} z(T_0) E[\hat{h}(T_0)] e^{-j2\pi f T_0} dT_0 = \int_{-\infty}^{\infty} z(T_0) h(T_0) e^{-j2\pi f T_0} dT_0$$

$$= \int_{-\infty}^{\infty} Z(u) H(f-u) du = H(f)$$

under the assumption that $H(f)$ is smooth (magnitude and phase) across the window bandwidth.

Note that $y(t) = (x(t) + w(t)) \ast h(t) = x(t) \ast h(t) + w(t) \ast h(t)$.

Since $x(t)$ is a Gaussian random process, then $x(t) \ast h(t)$ is a Gaussian random process because Gaussianity is preserved under linear transformations. Similarly, $w(t) \ast h(t)$ is a Gaussian random process. Also, the processes $x(t) \ast h(t)$ and $w(t) \ast h(t)$ are independent because $x(t)$ and $w(t)$ are independent. Hence $y(t)$ is a Gaussian random process with $E[y(t)] = 0$ and

$$\sigma_y^2 = E[y^2(t)] = \int_{-\infty}^{\infty} |H(f)|^2 \left( \varsigma_x(f)^2 + 1 \right) df$$
\[ \sigma^2_T \left| \mathcal{H} \right| = \mathbb{E} \left[ \mathbb{E} \left[ \hat{A}(T)^2 \right] \right] - \mathbb{E} \left[ \mathbb{E} \left[ \hat{A}(T)^2 \right] \right]^2 \]

\[ \mathbb{E} \left[ \mathbb{E} \left[ \hat{A}(T)^2 \right] \right] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} z(T_1) z(T_2) \mathbb{E} \left[ \mathbb{E} \left[ \hat{A}(T_1) \hat{A}(T_2) \right] \right] e^{-i\pi z(T_1)T_2 - i\pi z(T_2)T_1} dT_1 dT_2 \]

\[ \mathbb{E} \left[ \mathbb{E} \left[ \hat{A}(T)^2 \right]^2 \right] = \mathbb{E} \left[ \mathbb{E} \left[ \hat{A}(T)^2 \right] \right] \mathbb{E} \left[ \mathbb{E} \left[ \hat{A}(T)^2 \right] \right] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} z(T_1) z(T_2) \mathbb{E} \left[ \mathbb{E} \left[ \hat{A}(T_1) \hat{A}(T_2) \right] \right] e^{-i\pi z(T_1)T_2 - i\pi z(T_2)T_1} dT_1 dT_2 \]

so

\[ \sigma^2_T \left| \mathcal{H} \right| = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} z(T_1) z(T_2) \mathbb{E} \left[ \mathbb{E} \left[ \hat{A}(T_1) \hat{A}(T_2) \right] - h(T_1) h(T_2) \right] e^{-i\pi z(T_1)T_2 - i\pi z(T_2)T_1} dT_1 dT_2 \]

and

\[ \mathbb{E} \left[ \mathbb{E} \left[ \hat{A}(T_1) \hat{A}(T_2) \right] \right] = \frac{1}{T_2} \int_{-\pi/2}^{\pi/2} dt_1 \int_{-\pi/2}^{\pi/2} dt_2 \mathbb{E} \left[ \mathbb{E} \left[ y(t_1) w(T_1-T_2) y(t_2) w(T_2-T_2) \right] \right] \]

\[ = h(T_1) h(T_2) + R_y(t_1-t_2) \delta(t_1-t_2-T_1+T_2) \]

\[ \mathbb{E} \left[ \mathbb{E} \left[ \hat{A}(T_1) \hat{A}(T_2) \right] \right] = \frac{1}{T_2} \int_{-\pi/2}^{\pi/2} dt_1 \int_{-\pi/2}^{\pi/2} dt_2 h(T_1) h(T_2) \Rightarrow h(T_1) h(T_2) \]

\[ \Rightarrow \]

Now this is cancelled out by the mean

\[ \int_{-\pi/2}^{\pi/2} dt \int_{-\pi/2}^{\pi/2} dt \mathbb{E} \left[ \mathbb{E} \left[ y(t_1-t_2) \delta(t_1-t_2-T_1+T_2) \right] \right] \]

\[ = \int_{-\pi/2}^{\pi/2} dt \int_{-\pi/2}^{\pi/2} dt h(t_1-t_2) h(t_1-t_2+T_1) \]

\[ \Rightarrow \]

\[ \int_{-\pi/2}^{\pi/2} dt h(z+T_1) h(z+T_2) \]

\[ \Rightarrow \]

Something like a convolution. In Fourier domain will be

\[ \frac{1}{T} \mathbb{E} \left[ \mathbb{E} \left[ \hat{A}(T)^2 \right] \right] \sin(\frac{\pi z}{T}) \]
Thus:

\[ \sigma^2_{A(f)} = \int \int z(T_0)z(T_0') \left[ \frac{R_y^j(T_0-T_0')}{T^2} + \int \left( \frac{1-|z|^2}{T^2} \right) h(z+T_0)h(-z+T_0') \, dz \right] e^{-j2\pi u(T_0-T_0')} \, dT_0 \, dT_0' \]

\[ \left( \frac{T-|\tau_0-\tau_0'|}{T^2} \right) \int \left( S_x(u) + 1 \right) |H(u)|^2 \, e^{j2\pi u(T_0-T_0')} \, du \]

This term will lead to:

\[ (S_x(f)+1) |H(f)|^2 \int \left| Z_1(t) \right|^2 \, dt \]

\[ (I have to assume smoothness of S_x(f), also) \int \left| Z_2(t) \right|^2 \, dt \]

This term will lead to:

\[ |H(f)|^2 \int \left| Z_2(t) \right|^2 \, dt \]

where \( Z_1(t), Z_2(t) \) some composite tapers related to \( z(t) \)

So eventually I will have something like:

\[ \sigma^2_{A(f)} = |H(f)|^2 \left[ (S_x(f)+1) \frac{cw_1}{T} + \frac{cw_2}{T} \right] \]
c) We saw that \( E[\hat{H}(f)] = \int H(u)Z(f-u)\, du = H(f) \)

In order to have unbiased estimate \( \hat{H}(f) \) must have smooth magnitude and phase across the window bandwidth. If we pick a frequency \( f_0 \) we have:

\[
E[\hat{H}(f_0)] = \int_{f_0-BW/2}^{f_0+BW/2} H(u)Z(f_0-u)\, du
\]

\[
\approx \int |H(f)|e^{j\phi(f)}e^{-2\pi f f_0}Z(f)\, df
\]

\[
= |H(f_0)|e^{j\phi(f_0)}\int e^{j2\pi f f_0}Z(f)\, df
\]

we require \( 2\pi BW T_g(f_0) < \frac{\pi}{4} \Rightarrow BW < \frac{1}{4 T_g(f_0)} \) or more stringent \( BW = \frac{1}{4 \max[T_g(f)]} \), if in order for the phase of \( H(f) \) not to change significantly across the bandwidth of the window.

d) Consider the following structures:
Note that

\[
E\left[ h_{11}^2 (T_o) \right] = E\left[ \int_{-\frac{1}{2}}^{\frac{1}{2}} \left( y_{11}(t) + y_{12}(t) \right) W_1(t-T_o) \, dt \right] = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\infty}^{\infty} h_{11}(z) \left( x_{1}(t-z) + W_1(t) \right) \, dz \, W_1(t-T_o) + \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\infty}^{\infty} h_{21}(z) \left( x_{2}(t-z) + W_2(t-z) \right) \, dz \, W_1(t-T_o) = \int_{-\infty}^{\infty} h_{11}(z) \delta(T_o-z) \, dz = h_{11}(T_o)
\]

Similarly, \( E\left[ h_{12}^2 (T_o) \right] = h_{12}(T_o) \), \( E\left[ h_{21}^2 (T_o) \right] = h_{21}(T_o) \), \( E\left[ h_{22}^2 (T_o) \right] = h_{22}(T_o) \).

The variances can be computed as in part (b).
Problem 2

In this problem we compute spectral estimates based on data provided. The approach chosen in these solutions was to implement the segmented frequency averaging. We will see that breaking the data into segments helps reduce the variance as does frequency averaging. There is a trade-off however. If the segments are too small, the assumption of smoothness of the spectra is violated leading to large biases. A Hanning window has been chosen for use because it has desirable sidelobe properties. The max sidelobe level for a Hanning window is -32 dB (see Harris paper) as opposed to -13 dB for a rectangular window. This will help to prevent "leakage" if the data contains closely spaced tones.

One point that should be mentioned is no overlapping of the data snapshots was used. In practice, when windowing is used data segments are overlapped. It was not used here in order to keep the statistical results more traceable.

Statistics of Estimates

We want to compute the bias and the variance of $\hat{S}_x(f)$. The segment length is $T_L$. Freq. averaging is done across $N$ points,

$$\hat{S}_x(f) = \frac{1}{L} \sum_{i=1}^{L} h_n |X_w(f + i \frac{T_L}{L})|^2$$

where

$$X_w(t) = \int_{-\frac{T_L}{2}}^{\frac{T_L}{2}} w(t - \frac{T_L}{2}) \cdot x(t) e^{-j2\pi ft} \, dt$$
where $w(t)$ is the Hamming window of duration $T_c$ centered at $T_c$ for each segment.

\[
E\left[\hat{S}_x(t)\right] = \frac{1}{T_c} \sum_{n=-N}^{N} h_n \mathbb{E}\left[|X_n(t + \frac{n}{T_c})|^2\right] \\
= \frac{1}{T_c} \sum_{n=-N}^{N} h_n \int_{T_c/2}^{T_c+T_c/2} \int_{T_c/2}^{T_c+T_c/2} E\left[X(t_1)X(t_2)\right] W(t_1-T_c) W(t_2-T_c) e^{-j\frac{2\pi}{T_c}(t_1-t_2)} dt_1 dt_2 \\
R_x(t_1-t_2) \rightarrow \text{assuming WSS process} \\
\int_{-\infty}^{\infty} S_x(u) e^{j2\pi u (t_1-t_2)} du
\]

Invoking smoothness over freq. band of averaging we have,

\[
E\left[\hat{S}_x(t)\right] = \frac{1}{T_c} \sum_{n=-N}^{N} h_n S_x(t) \int W(t + \frac{n}{T_c} - u) W^*(t + \frac{n}{T_c} - u) du
\]

Observe that the phase shifts due $T_c$ window delays are cancelled out.

\[
= \frac{1}{T_c} \sum_{n=-N}^{N} h_n S_x(t) \int_{T_c/2}^{T_c+T_c/2} |w(t)|^2 dt \\
= S_x(t) \sum_{n=-N}^{N} h_n \frac{c_w}{T_c} \quad (c_w = 1.5 \text{ from Harris paper})
\]

$S_x$ for unbiased estimate assuming constant $h_n = h$ we have $(2N+1) h \frac{1.5}{T_c} = 1 \Rightarrow h = \frac{T_c}{3.5 (2N+1)}$

Note that we assumed $S_x(t)$ smooth over $(2N+1) \frac{2}{T_c}$

To compute the variance of $S_x(t)$ we use the key result for Gaussian processes:

\[
\text{var}_{\text{real}} = \left(\text{var}_{\text{imag}}\right)^2 = \left(\text{var}_{\text{real}}\right)^2 \quad \text{conditioned we are away from DC i.e. } |t| + \frac{N}{T_c} \rightarrow B_w/2
\]

(Close to DC the variance is higher)
Similar to lecture notes we have:

\[ \sigma_x^2 = \frac{1}{L^2} \sum_{i=1}^{L} \sum_{j=1}^{L} \sum_{n=-N}^{N} \sum_{m=-m}^{m} h_n h_m \int_{-\infty}^{\infty} W(u) W(u - \frac{n \cdot \Delta u}{T_c}) e^{j2\pi(u \cdot \Delta u)} du \]

Define

\[ P_{ij}^{mn} = \int_{-\infty}^{\infty} W(u) W(u - \frac{n \cdot \Delta u}{T_c}) e^{j2\pi u \cdot (T_i - T_j)} du \]

\[ = \int_{-\infty}^{\infty} |W(u)|^2 du \]

Note that \(|P_{mn}^{ij}| < 1\).

\[ \frac{\sigma_x^2}{S^2(\Delta t)} \] can be written in matrix form as follows:

\[ \frac{\sigma_x^2}{S^2(\Delta t)} = \frac{1}{L^2} \begin{bmatrix} |P_{11}^{11}|^2 & \ldots & |P_{11}^{N1}|^2 \\ \vdots & \ddots & \vdots \\ |P_{N1}^{11}|^2 & \ldots & |P_{N1}^{N1}|^2 \end{bmatrix} H K H^T \]

where \(H^T K H\) is a vector of length \(2N+1\) whose elements are \(h_n^2\).

If \(h_n = h = \frac{1}{T_c} \) then \(\mathbb{E} h_n = \frac{1}{T_c}\) and \(h_n^2 = \frac{1}{2N+1}\).

\[ |P_{mn}^{ij}| = \int_{-T_c/2}^{T_c/2} W(u) W(u - \frac{n \cdot \Delta u}{T_c}) du \]

\[ = \frac{T_c}{\Delta u} \int_{-T_c/2}^{T_c/2} |W(u)|^2 e^{j2\pi \frac{u \cdot (y_1 - y_2)}{T_c}} du \]

For the Hann window:

\[ w(t) = \frac{2}{T_c} \cos^2 \left( \frac{\pi t}{T_c} \right) \quad |t| \leq T_c/2 \]
\[ W(f) = \frac{1}{T} \int \left\{ \frac{1}{2} + \frac{1}{2} \cos \left( \frac{\pi}{T} f \right) \right\} = \frac{2}{T} \left[ \delta \left( \frac{f}{2} \right) + \delta \left( \frac{f - \frac{2\pi}{T}}{2} \right) + \delta \left( \frac{f + \frac{2\pi}{T}}{2} \right) \right] \]

\[ \Rightarrow W(f) = \text{sinc} \left( \frac{\pi f}{2} \right) + \frac{1}{2} \text{sinc} \left( \frac{\pi f}{2} - \pi \right) + \frac{1}{2} \text{sinc} \left( \frac{\pi f}{2} + \pi \right) \]

Note that the width of the main lobe is \( \frac{1}{T} \), so if \( n \cdot \frac{1}{T} > \frac{2}{T} \) then the correlation goes to 0.

If \( n \cdot \frac{1}{T} \) then \( \int_{-\infty}^{\infty} W(u) W^*(u - \frac{1}{T}) \, du = 1 \)

\[ \text{CWT} \]

If \( |n - \frac{1}{T}| = 1 \) \( \Rightarrow \int_{-\infty}^{\infty} W(u) W^*(u - \frac{1}{T}) \, du = \frac{2}{3} \)

\[ \text{CWT} \]

If \( |n - \frac{1}{T}| = 2 \) \( \Rightarrow P_{nn_2} = \frac{1}{6} \)

If \( |n - \frac{1}{T}| > 2 \) \( \Rightarrow P_{nn_2} = 0 \)

Note that for frequency averaging the min spacing is \( \frac{1}{T} \).

It is easy to compute the matrix \([P_{nn_2}]^{(1)}\).

\[
\begin{bmatrix}
p_{ii} & 1 & 0 \\
0 & \frac{1}{3} & 0 \\
0 & 0 & \frac{1}{3} \\
\end{bmatrix}
\]

\[ 2NH \times 2NH \]

and \( \frac{1}{2N} h^T \left[ \left[ P_{nn_2} \right]^{(1)} \right] h' = \frac{1}{L^2 (2NH)^2} \left[ (2NH) \left( \frac{2}{3} \right) + (2N) \left( \frac{2}{3} \right) + (2N) \left( \frac{2}{3} \right) \right] \]

It easy to conclude that \( \sigma^2 \frac{\delta(f)}{L^2} \)

\( \delta(f) \) is a real function so no consideration must be given to the phase of the estimate. It the spectrum appears
to have narrowband phenomena then we need to adjust
the segment length is order not to violate the smoothness
assumption. In our case, there exist two tonals at
5 Hz and 10 Hz and a narrowband component at 40 Hz.

Figure 1 shows the estimate using no averaging (N=0)
in order to see the tonals clearly.

Figure 2 shows the estimate without averaging but
increased segments (Tseg = 40 sec). We note that the
variance is increased relative to figure 1 because we
do less segment averaging.

Figure 3 shows the estimate with N=3 but reduced segments
(Tseg = 5 sec). The variance is reduced at the
expense of resolution.
function psd_fft

load data;
% column vector of data
fs=1000;
% sampling frequency of data in Hz
T=1000;
% duration of data in sec -> fundamental freq. limitation to 1/T Hz.
N=1;
% averaging in freq. (total 2N+1 points)
Tseg=5;
% segment of data in sec

w=(2/Tseg)*cos(pi*linspace(-Tseg/2,Tseg/2,Tseg*fs)'./Tseg).^2;
% use Hanning window
Cw=1.5/Tseg;
% from Harris paper
h=ones(1,2*N+1)/Cw/(2*N+1);
% condition for unbiased estimate

Nfft=2^nextpow2(fs*Tseg);
% fft points

df=fs/Nfft;
% freq. resolution for plotting
res=4/Tseg;
% mainlobe width for Hanning
M=ceil(res/df);
% min spacing to average uncorrelated r.v.

S=zeros(Nfft,1);
K=floor(T/Tseg);
% loop number
for i=1:K
seg=data(1+(i-1)*fs*Tseg:i*fs*Tseg);
% take one segment of data at a time
Xw=fftshift(fft(w.*seg,Nfft));
Y=conj(Xw).*Xw;
Y=[zeros(N*M,1); Y; zeros(N*M,1)];
% append zeros to account for averaging at the limits
    for k=1:Nfft
        S(k)=S(k) + h*Y(k:M:2*N*M+k)/K;
    end
end

plot([-0.5:1/Nfft:.5-1/Nfft]*fs,10*log10(S/fs^2));
% normalization for estimate
title('Figure 3, N=1, Tseg=5sec') ,xlabel('f (Hz)'), ylabel('dB')