Quiz #3 solutions.

Problem 1

i) $\hat{S}_x(f) = \dfrac{|X_w(f)|^2}{T_L}$ is our spectral estimation function compute the mean:

$$E[\hat{S}_x(f)] = \frac{E\{|X_w(f)|^2\}}{T_L} =$$

$$= \frac{1}{T_L}\int_{-T_L/2}^{T_L/2} dt_1 \int_{-T_L/2}^{T_L/2} dt_2 \; w(t_1)w(t_2)\underbrace{E[x(t_1)x^*(t_2)]}_{R_x(t_1-t_2) = \int S_x(u)e^{j2\pi u(t_1-t_2)}du} e^{-j2\pi f(t_1-t_2)}$$

$$= \frac{1}{T_L}\int_{-\infty}^{\infty} S_x(u) \int_{-T_L/2}^{T_L/2} w(t_1)e^{-j2\pi(f-u)t_1}dt_1 \int_{-T_L/2}^{T_L/2} w(t_2)e^{-j2\pi(u-f)t_2}dt_2$$

$$= \frac{1}{T_L}\int_{-\infty}^{\infty} S_x(u)\, \bar{W}(f-u)\,\bar{W}(u-f)\,du \overset{w(t)\text{ real} \Rightarrow \bar{W}(f) = \bar{W}^*(-f)}{=}$$

$$= \frac{1}{T_L}\int_{-\infty}^{\infty} S_x(u)\,|\bar{W}(f-u)|^2 du$$

assuming that $S_x(f)$ is smooth across the bandwidth of $\bar{W}(f)$ then

$$E[\hat{S}_x(f)] = \frac{1}{T_L} S_x(f)\int_{-\infty}^{\infty} |\bar{W}(u)|^2 du$$

for unbiased estimate we require

$$\int_{-\infty}^{\infty} |\bar{W}(u)|^2 du = \int_{-T_L/2}^{T_L/2} w^2(t)\,dt = T_L$$

For the Hanning taper I require

$$\int_{-T_L/2}^{T_L/2} a^2 \cos^4\left(\frac{\pi t}{T_L}\right) dt = T_L \Rightarrow$$

$$\int_{-T_L/2}^{T_L/2} \frac{a^2}{4}\left(1+\cos\left(\frac{2\pi t}{T_L}\right)\right)^2 dt = \frac{a^2}{4}\int_{-T_L/2}^{T_L/2} 1+\cos^2\left(\frac{2\pi t}{T_L}\right)+\underbrace{2\cos\left(\frac{2\pi t}{T_L}\right)}_{\int dt = 0} dt$$

$$= \frac{a^2}{4}\left[T_L + \frac{1}{2}\int_{-T_L/2}^{T_L/2} 1 + \underbrace{\cos\left(\frac{4\pi t}{T_L}\right)}_{\int dt = 0} dt\right] = \frac{a^2}{4}\left(T_L + \frac{T_L}{2}\right)$$

$$= \frac{a^2}{4}\frac{3\cancel{T_L}}{2} = \cancel{T_L} \Rightarrow a = \sqrt{\frac{8}{3}}$$

ii) $$\sigma^2_{\hat{S}_x(f)} = E\left[|\hat{S}_x(f)|^2\right] - \left|E\left[\hat{S}_x(f)\right]\right|^2$$

$$E\left[|\hat{S}_x(f)|^2\right] = E\left[\frac{|X_w(f)|^4}{T_L^2}\right]$$

since $X_w(f)$ is complex Gaussian r.v. $\forall f$ with zero mean we can invoke the product moment thm for Gaussians so

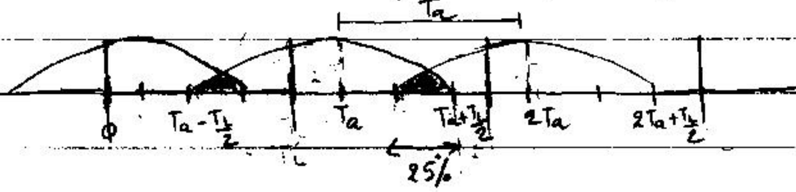
$$E\left[\frac{|X_w(f)|^4}{T_L^2}\right] = \frac{2}{T_L^2}E\left[|X_w(f)|^2\right]^2$$

Hence, $$\sigma^2_{\hat{S}_x(f)} = 2\frac{E\left[|X_w(f)|^2\right]^2}{T_L^2} - \left(\frac{E\left[|X_w(f)|^2\right]}{T_L}\right)^2$$

$$= \frac{E\left[|X_w(f)|^2\right]^2}{T_L^2} \overset{\text{part (i)}}{=} \frac{S_x^2(f)\,\cancel{T_L^2}}{\cancel{T_L^2}} = S_x^2(f)$$

Thus windowing doesn't mitigate the variance.

iii) Below is an example of 25% overlap



no overap $\Rightarrow$ $T_a = T_L$

with overlap $\Rightarrow$ $A = T_a + \frac{T_L}{2} - \left(2T_a - \frac{T_L}{2}\right) = T_L - T_a =$

$= T_L\left(1 - T_a/T_L\right) = T_L(1 - x)$

Compute : $E\left[\hat{S}_x^n(f)\,\hat{S}_x^m(f)\right] =$

$\frac{1}{T_L^2} E\left[X_w^n(f)\,X_w^{n*}(f)\,X_w^m(f)\,X_w^{m*}(f)\right]$ $\underline{\text{product}}$ moment thm

$= \frac{1}{T_L^2}\left[E\left[X_w^n(f)X_w^{n*}(f)\right]E\left[X_w^m(f)X_w^{m*}(f)\right] + E\left[X_w^n(f)X_w^{m*}(f)\right]E\left[X_w^{n*}(f)X_w^m(f)\right]\right]$

$$= \left[\underbrace{\frac{E\left[|X_w^n(f)|^2\right]}{T_L}\,\frac{E\left[|X_w^m(f)|^2\right]}{T_L}}_{S_x^2(f)} + \frac{\left|E\left[X_w^n(f)\,X_w^{m*}(f)\right]\right|^2}{T_L^2}\right] \quad (1)$$

need to know: $E\left[X_w^n(f)\,X_w^{m*}(f)\right] =$

$$\int_{nT_a - T_L/2}^{nT_a + T_L/2} dt_1 \int_{mT_a - T_L/2}^{mT_a + T_L/2} dt_2\; w(t_1 - nT_a)\,w(t_2 - mT_a)\,\underbrace{E\left[x(t_1)x^*(t_2)\right]}_{R_x(t_1 - t_2) = \int_{-\infty}^{\infty} S_x(u) e^{j2\pi u(t_1 - t_2)}du}\,e^{-j2\pi f(t_1 - t_2)}$$

$$= \int_{-\infty}^{\infty} S_x(u) \int_{nT_a - T_L/2}^{nT_a + T_L/2} w(t_1 - nT_a)\,e^{-j2\pi t_1(f-u)}\,dt_1 \int_{mT_a - T_L/2}^{mT_a + T_L/2} w(t_2 - mT_a)\,e^{-j2\pi t_2(u-f)}\,dt_2$$

$$= \int_{-\infty}^{\infty} S_x(u)\,\bar{W}(f-u)\,e^{-j2\pi(f-u)nT_a}\,\bar{W}(u-f)\,e^{-j2\pi(u-f)mT_a}\,du$$

$$= \int_{-\infty}^{\infty} S_x(u)\,|\bar{W}(f-u)|^2\,e^{-j2\pi(f-u)T_a(n-m)}\,du \qquad \underline{\underline{S_x(f)\ \text{"smooth"}}}$$

$$= S_x(f)\int_{-\infty}^{\infty} |\bar{W}(u)|^2\,e^{-j2\pi u T_a(n-m)}\,du$$

I want to compute the variance of the estimate

$$\hat{S}_x(f) = \frac{1}{L}\sum_{n=1}^{L} \hat{S}_x^n(f)$$

$$\sigma^2_{\hat{S}_x(f)} = \frac{1}{L^2} E\left[\left(\sum_{n=1}^{L} \hat{S}_x^n(f)\right)^2\right] - \frac{1}{L^2} E\left[\sum_{n=1}^{L} \hat{S}_x^n(f)\right]^2$$

$$= \frac{1}{L^2}\sum_{n=1}^{L}\sum_{m=1}^{L} E\left[\hat{S}_x^n(f)\,\hat{S}_x^m(f)\right] - \frac{1}{L^2}\left(\sum_{n=1}^{L} \underbrace{E\left[\hat{S}_x^n(f)\right]}_{S_x(f)}\right)^2$$

eq.①

$$= \cancel{\frac{1}{L^2}\sum_n\sum_m S_x^2(f)} + \frac{1}{L^2}\sum_n\sum_m \frac{1}{T_L^2}\left|E\left[X_w^n(f)X_w^{m*}(f)\right]\right|^2 - \cancel{S_x^2(f)}$$

$$\Rightarrow \sigma^2_{\hat{S}_x(f)} = \frac{1}{L^2}\frac{1}{T_L^2}\sum_n\sum_m S_x^2(f)\left|\int_{-\infty}^{\infty} |\bar{W}(u)|^2 e^{-j2\pi u T_a(n-m)}\,du\right|^2$$

If we define $p(n-m) \triangleq \frac{1}{T_L}\int_{-\infty}^{\infty} |\bar{W}(u)|^2 e^{-j2\pi u T_a(n-m)}\,du$

$T_L = \int_{-\infty}^{\infty}|\bar{W}(u)|^2 du$ for unbiased estimate

then

$$p(n-m) \triangleq \frac{\int_{-\infty}^{\infty} |W(u)|^2 e^{-j2\pi T_a(n-m)}\,du}{\int_{-\infty}^{\infty} |\bar{W}(u)|^2\,du}$$

Note that $|p(x)| \le 1$ from Cauchy-Schwarz inequality. Also we assume that for $|n-m|>1 \Rightarrow p(x)=0$ because $T_L \gg$ correlation scale of $x(t)$ so segments that do not overlap are uncorrelated in time (this assumption can be tricky in practice)

Hence I can write $\frac{\sigma^2}{S_x^2(f)}$ in matrix form

$$\frac{\sigma^2}{S_x^2(f)} = \frac{1}{L^2}\begin{bmatrix}1\\ \vdots \\ \vdots \\ 1\end{bmatrix}^T \begin{bmatrix}1 & |p(x)|^2 & 0 & 0\\ |p(x)|^2 & \ddots & \ddots & \\ 0 & \ddots & \ddots & \\ 0 & & & 1\end{bmatrix}\begin{bmatrix}1\\ \vdots \\ \vdots \\ 1\end{bmatrix}$$

$$\text{so } \frac{\sigma^2}{S_x^2(f)} = \frac{1}{L^2}\left(L \cdot 1 + 2(L-1)\,|p(x)|^2\right)$$

$$\overset{L \gg 1}{\cong} \frac{1}{L^2}\left(L + 2L\,|p(x)|^2\right) = \frac{1+2|p(x)|^2}{L}$$

$$\overset{|p(x)|^2=\cos^2(\frac{\pi}{2}x)}{=} \frac{1}{L}\left(1 + 2\cos^2\left(\frac{\pi}{2}x\right)\right) \overset{L=\frac{T}{T_L \cdot x}}{=} \frac{x\,T_L}{T}\left(1 + 2\cos^2\left(\frac{\pi x}{2}\right)\right)$$

I want to minimize $\frac{\sigma^2}{S_x^2(f)}$ so

$$\frac{d\, \frac{xT_L}{T}\left(1 + 2\cos^2(\frac{\pi}{2}x)\right)}{dx} = 0$$

$$\Rightarrow \frac{T_L}{\cancel{T}}\left(1 + 2\cos^2\left(\frac{\pi}{2}x\right)\right) - \frac{x\cdot T_L}{\cancel{T}}\left(4\cos\left(\frac{\pi}{2}x\right)\sin\left(\frac{\pi}{2}x\right)\frac{\pi}{2}\right) = 0$$

$$\Rightarrow 1 + 2\cos^2\left(\frac{\pi}{2}x\right) - \pi x \sin(\pi x) = 0 \Rightarrow$$

$$\boxed{2 + \cos(\pi x) = \pi x \sin(\pi x)}$$

The above equation determines optimum overlap for minimizing the variance.

Problem 2

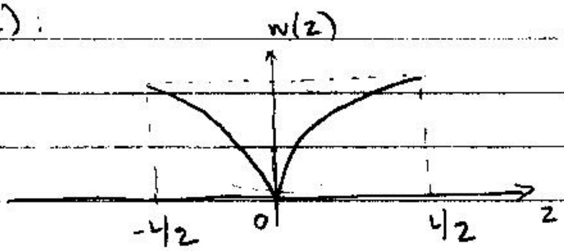
i) From the MRA constraint we have:

$$\int_{-L/2}^{L/2} w(z)\,dz = 1 \qquad (\text{I am assuming broadside steering})$$

$$\text{so} \int_{-L/2}^{L/2} a \sin^2 \frac{\pi z}{L}\, dz = \frac{a}{2}\int_{-L/2}^{L/2} 1 - \underbrace{\cos\frac{2\pi z}{L}}_{\int dz \to 0}\, dz =$$

$$= \frac{aL}{2} = 1 \Rightarrow \boxed{a = \frac{2}{L}}$$

Plot of $w(z)$:



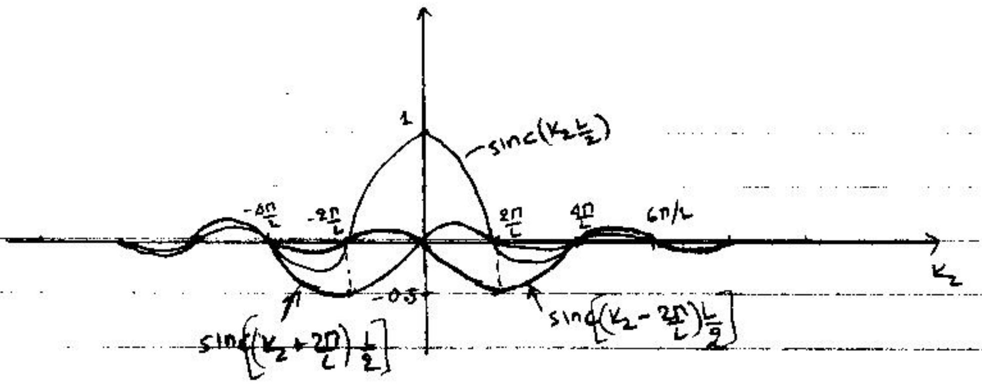
Frequency wavenumber response function:

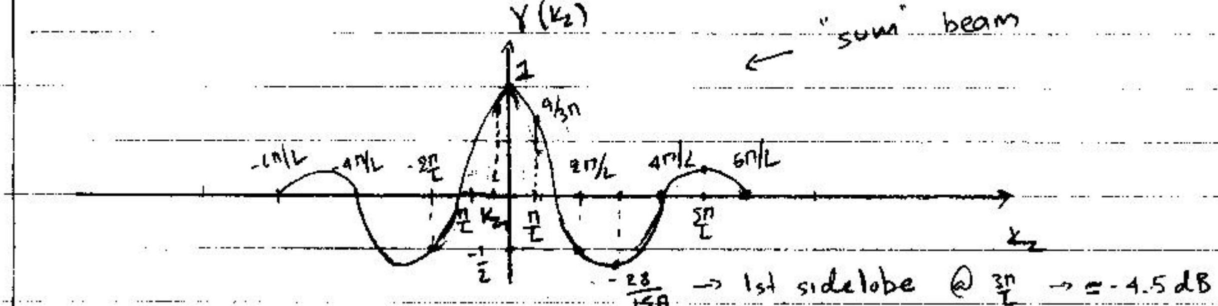
$$\Upsilon(k_z) = \int_{-L/2}^{L/2} w(z) e^{-jk_z z}\, dz = \int_{-L/2}^{L/2} a \sin^2 \frac{\pi z}{L} e^{-jk_z z}\, dz$$

$$= \frac{a}{2}\int_{-L/2}^{L/2} e^{-jk_z z}\, dz - \frac{a}{2}\int_{-L/2}^{L/2} \cos\frac{2\pi z}{L} e^{-jk_z z}\, dz$$

$$= \frac{a}{2} L \operatorname{sinc}\left(k_z \frac{L}{2}\right) - \frac{a}{2}\frac{1}{2\pi}\left[L \operatorname{sinc}\left(\frac{k_z L}{2}\right) * \left[\pi\delta\left(k_z - \frac{2\pi}{L}\right) + \pi\delta\left(k_z + \frac{2\pi}{L}\right)\right]\right]$$

$$\overset{aL=2}{=} \operatorname{sinc}\left(k_z \frac{L}{2}\right) - \frac{1}{2}\operatorname{sinc}\left[\left(k_z - \frac{2\pi}{L}\right)\frac{L}{2}\right] - \frac{1}{2}\operatorname{sinc}\left(k_z + \frac{2\pi}{L}\right)\frac{L}{2}$$





Note that the mainlobe is narrower than that of a uniform weighting aperture at the expense of higher sidelobes.

ii) $\int_{-L/2}^{L/2} w^2(z)\,dz = \int_{-L/2}^{L/2} a^2 \sin^4 \frac{\pi z}{L}\,dz = \frac{a^2}{4}\int_{-L/2}^{L/2} \left(1-\cos(2\pi z/L)\right)^2 dz$

$$= \frac{a^2}{4}\int_{-L/2}^{L/2} 1 + \cos^2\left(\frac{2\pi z}{L}\right) - \underbrace{2\cos\left(\frac{2\pi z}{L}\right)}_{\int dz = 0}\,dz =$$

$$= \frac{a^2}{4}L + \frac{a^2}{8}\int_{L/2}^{L/2} 1 - \underbrace{\cos\left(\frac{4\pi z}{L}\right)}_{\int dz = 0}\,dz = \frac{a^2}{4}L + \frac{a^2}{8}L$$

$$= \frac{3a^2L}{8} = \frac{3}{2L}$$

Hence $NA_w = \frac{1}{L\,\frac{3}{2L}} = \frac{2}{3} < 1$ (for uniform weighting)

The Hanning taper is:

$$h(z) = \begin{cases} a\cos^2\left(\frac{\pi z}{L}\right) & |z| \le \frac{L}{2} \\ 0 & |z| > L/2 \end{cases}$$

Similarly we can prove that $a = \frac{2}{L}$ (MRA condition) and $\int_{-L/2}^{L/2} h^2(z)\,dz = \frac{3}{2L}$ resulting the same noise enhancement. Obviously, tapers having the same $\int_{-L/2}^{L/2} w^2(z)\,dz$ will yield the same NA_w.

iii) In this case the frequency-wavenumber response function is:

$$Y(k_z) = -\int_{-L/2}^{0} \sin^2\left(\frac{\pi z}{L}\right) e^{-jk_z z}\,dz + \int_{0}^{L/2} \sin^2\left(\frac{\pi z}{L}\right) e^{-jk_z z}\,dz$$

$$= -\frac{1}{2}\int_{-L/2}^{0}\left(1-\cos\frac{2\pi z}{L}\right)e^{-jk_z z}\,dz + \frac{1}{2}\int_{0}^{L/2}\left(1-\cos\left(\frac{2\pi z}{L}\right)\right)e^{-jk_z z}\,dz$$

$$= -\frac{1}{2}\int_{0}^{L/2}\left(1-\cos\frac{2\pi z}{L}\right)e^{jk_z z}\,dz + \frac{1}{2}\int_{0}^{L/2} 1-\cos\left(\frac{2\pi z}{L}\right)e^{-jk_z z}\,dz$$

$\underline{\underline{a - a^* = 2j\,\mathrm{Im}(a)}}$

$$2j\,\mathrm{Im}\left[\frac{1}{2}\int_{0}^{L/2}\left(1-\cos\left(\frac{2\pi z}{L}\right)\right)e^{-jk_z z}\,dz\right]$$

$$= j I_m \left[\text{F.T.}\left[\sqcap_{0}^{L/2} \right] - \text{F.T.}\left[\sqcap_{0}^{L/2} \cdot \cos\left(\frac{2\pi z}{L}\right) \right] \right]$$

$$= j I_m \left[e^{-jk_z L/4} \frac{L}{2} \operatorname{sinc}\left(\frac{k_z L}{4}\right) - \frac{e^{-jk_z L/4}}{2\pi} \frac{L}{2} \operatorname{sinc}\left(\frac{k_z L}{4}\right) * \left(\pi\delta\left(k_z - \frac{2\pi}{L}\right) + \pi\delta\left(k_z + \frac{2\pi}{L}\right)\right) \right]$$

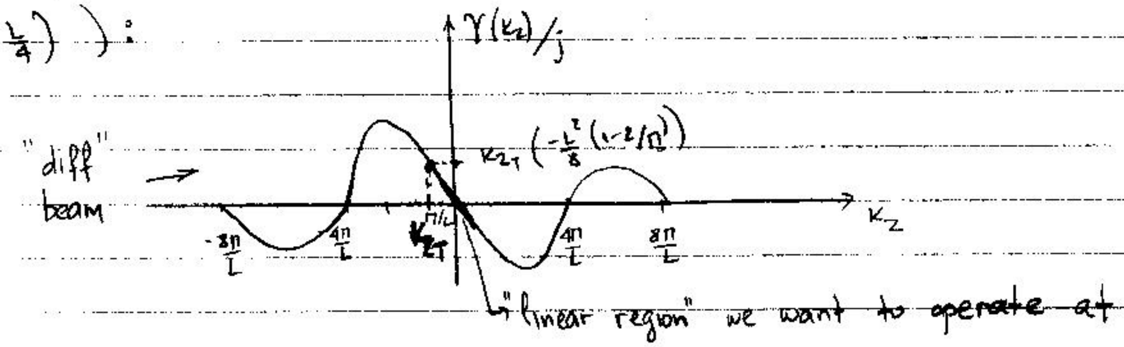
$$= j \sin\left(-k_z \frac{L}{4}\right) \left[\frac{L}{2} \operatorname{sinc}\left(\frac{k_z L}{4}\right) - \frac{L}{4} \operatorname{sinc}\left(\left(k_z - \frac{2\pi}{L}\right)\frac{L}{4}\right) - \frac{L}{4} \operatorname{sinc}\left(\left(k_z + \frac{2\pi}{L}\right)\frac{L}{4}\right) \right]$$

$$= -j \frac{L}{2} \sin\left(k_z \frac{L}{4}\right) \left[\operatorname{sinc}\left(k_z \frac{L}{4}\right) - \frac{1}{2} \operatorname{sinc}\left(\left(k_z - \frac{2\pi}{L}\right)\frac{L}{4}\right) - \frac{1}{2} \operatorname{sinc}\left(\left(k_z + \frac{2\pi}{L}\right)\frac{L}{4}\right) \right]$$

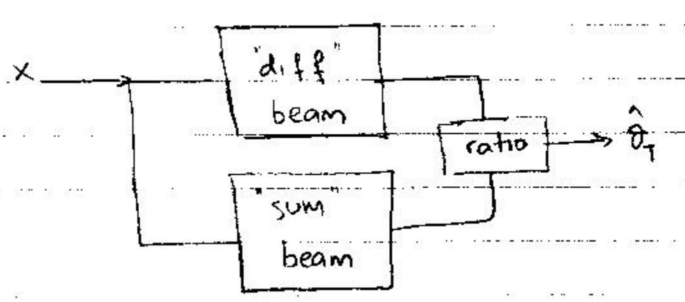
Note that $\left.\dfrac{d\Upsilon(k_z)}{dk_z}\right|_{k_z=0} = -j \dfrac{L}{2}\dfrac{L}{4}\left(1 - \dfrac{1}{2}\dfrac{1}{\frac{\pi}{2}} - \dfrac{1}{2}\dfrac{1}{\frac{\pi}{2}}\right) =$

$$= -j \frac{L^2}{8}\left(1 - \frac{2}{\pi}\right)$$

so the slope is negative which means that the plot of $\Upsilon(k_z)$ looks like (odd function due to $\sin\left(k_z \frac{L}{4}\right)$):



Consider the following structure:



This structure estimates the angle of arrival ϑ_T of a target measured off broadside without electronically steering.

Suppose that ϑ_T small $\Rightarrow$ $\sin\vartheta_T \simeq \vartheta_T$ $\Rightarrow$

$K_{z_T} = \frac{2\pi}{\lambda}\sin\vartheta_T \simeq \frac{2\pi}{\lambda}\vartheta_T$. The "diff" beam imposes a gain of $K_{z_T} \cdot \left.\frac{d\Upsilon(k_z)}{dk_z}\right|_{k_z=0}$ to the incoming signal

assuming we operate in the linear region

while the "sum" beam imposes a gain $\simeq$ 0 dB. Forming the ratio the unknown amplitude is cancelled out thus computing the angle of arrival:

$$\text{Ratio} = \frac{K_{z_T}\cdot\left(-\frac{L^2}{8}\left(1-\frac{2}{\pi}\right)\right)\cdot\cancel{x}}{1\cdot\cancel{x}} \approx$$

$$\frac{2\pi}{\lambda}\,\vartheta_T\left(-\frac{L^2}{8}\left(1-\frac{2}{\pi}\right)\right) \Rightarrow \vartheta_T \text{ is known.}$$

In order to assume that the "sum" beam puts no gain on x

note that we are in the "linear region" for this condition

$K_{z_T} < \frac{\pi}{L} \Rightarrow \frac{2\pi}{\lambda}\sin\vartheta_T < \frac{\pi}{L} \Rightarrow \sin\vartheta_T < \frac{\lambda}{2L}$

$\approx \boxed{\vartheta_T < \frac{\lambda}{2L}}$ (half the broadside resolution) of a uniformly weighted aperture