The statistics of ocean-acoustic ambient noise

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Abstract
With the assumption that the ocean-acoustic ambient noise field is a random process that obeys the central limit theorem, many useful statistical properties of subsequent intensity and mutual intensity measurements are readily deduced by respective applications of coherence theory and estimation theory.

1. Introduction
The temporal and spatial coherence of ocean-acoustic ambient noise is analyzed from a statistical perspective. In keeping with experimental observations in both deep and shallow water, where a large number of independent sea surface source contributions are summed together by the ocean waveguide, the spatial ambient noise field is taken to effectively behave as a multidimensional circular complex Gaussian random process. From this underlying Gaussianity, some interesting and useful deductions can be made about the statistical properties of typical ambient noise measurements. For example, analytic expressions are derived for the coherence time of the noise process, and the degrees of freedom in a measurement of the noise field's intensity and mutual intensity are expressed as a function of measurement time and temporal coherence. Analytic expressions are then obtained for both the joint and individual distributions of finite time-averaged mutual-intensity as formed by the sample covariance of a spatial array of noise field measurements. Finally, a quantitative expression is derived for the amount of information that can be inferred from noise field measurements, and general expressions for the optimal resolution of the environmental parameters upon which the noise depends are presented.

2. Ambient noise field statistics
The vector $\mathbf{\Phi}[n]$ contains the instantaneous circular complex Gaussian random noise fields $\phi_i\ [n]$ measured by sensors i=1,2,3 ...,N_Φ at discrete time n. To permit spatial coherence across the array at any instant n, it is assumed that any sample $\phi_i[n]$ may be correlated with

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any $\phi_j[n]$. However, all field samples with differing discrete-time indexes are assumed to be independent so that they obey

$$P(\mathbf{\Phi}[1],\mathbf{\Phi}[2],\mathbf{\Phi}[3]\ldots\mathbf{\Phi}[N]) = \prod_{n=1}^{N} \{\pi^{N_\Phi}|\mathbf{M}|\}^{-1} \exp\{-\mathbf{\Phi}[n]^{\mathrm{H}}\mathbf{M}^{-1}\mathbf{\Phi}[n]\}, \qquad (1)$$

where the covariance is the Hermitian matrix $\mathbf{M}=\langle\mathbf{\Phi}[n]\mathbf{\Phi}^{\mathrm{H}}[n]\rangle$.

The complex sample covariance

$$\mathbf{S}[N] = \frac{1}{N}\sum_{n=1}^{N}\mathbf{\Phi}[n]\mathbf{\Phi}^{\mathrm{H}}[n], \qquad (2)$$

is then a sufficient statistic for estimation of **M**. Here, the instantaneous mutual intensity samples $\mathbf{\Phi}[n]\mathbf{\Phi}^{\mathrm{H}}[n]$ are assumed to be identically distributed over time index n. The expectation value of the sample covariance is **M**, and the covariance of the sample covariance is $\mathbf{C}=(1/N)\mathbf{M}\otimes\mathbf{M}^*$, where $\otimes$ denotes the Kronecker product, so that the elements of **C** are fourth order tensors.

2. Sample size as a function of time and coherence

An expression for the maximum number of independent samples available in a stationary measurement period is now derived. This is given in terms of the temporal coherence of the received field and the measurement time. In loose terms, the concept is to determine the number of times the received field is expected to fluctuate independently during the given measurement period. This is achieved by inspection of the signal-to-noise ratio (SNR) of the measurement. Here the SNR is defined as the squared-mean to variance ratio. For the discretely sampled case, the SNR of a sample covariance element is

$$\mathrm{SNR}\{S_{ij}[N]\} = \frac{|M_{ij}|^2}{C_{ij,ij}} = N\frac{|M_{ij}|^2}{M_{ii}M_{jj}}, \qquad (3)$$

where M_{ii} is positive semi-definite and equal to the expected intensity at sensor i. Here the number of independent samples N is equal to the SNR for a diagonal element of the sample covariance, such that $\mathrm{SNR}\{S_{ii}[N]\} = N$. This is because $S_{ii}[1]$ has an expectation value that equals its standard deviation under the CCGR field assumption, and all N samples are independent and identically distributed.

Analogously, the number of independent samples available in a continuous time measurement of S_{ij} is given by its SNR. To show this, the sample covariance of Eq (1) can be equivalently written as a continuous temporal average

$$\mathbf{S}(T) = \frac{1}{T}\int_{-T/2}^{T/2} \mathbf{\Phi}(t)\mathbf{\Phi}^{\mathrm{H}}(t)dt. \tag{4}$$

For the continuous case, the SNR of $S_{ij}(T)$ is defined as

$$\mathrm{SNR}\{S_{ij}(T)\} = \frac{< R_{ij}(T) >^2 + < I_{ij}(T) >^2}{\sigma^2_{R_{ij}(T)} + \sigma^2_{I_{ij}(T)}}, \tag{5}$$

where $R_{ij}(T)=\mathrm{Re}\{S_{ij}(T)\}$, $I_{ij}(T)=\mathrm{Im}\{S_{ij}(T)\}$, $\sigma^2_{R_{ij}(T)}$ is the variance of $R_{ij}(T)$, and $\sigma^2_{I_{ij}(T)}$ is the variance of $I_{ij}(T)$. It is not difficult to show that $<R_{ij}(T)>=\mathrm{Re}\{M_{ij}\}$ and $<I_{ij}(T)> =\mathrm{Im}\{M_{ij}\}$.

Expressions for the variances can also be obtained, but with more difficulty. First, it is useful to employ some definitions from statistical optics. The complex degree of coherence is defined as

$$\gamma_{ij}(\tau)=<\phi_i(t+\tau)\phi^*_j(t)>/(M_{ii}M_{jj})^{1/2}, \tag{6}$$

and the complex coherence factor is defined as

$$\nu_{ij}=\gamma_{ij}(0)=M_{ij}/(M_{ii}M_{jj})^{1/2}. \tag{7}$$

By defining the normalized cross-spectral density $\mathrm{S}_{ij}(f)$ as the Fourier transform of the complex degree of coherence $\gamma_{ij}(\tau)$, and $Q_{ij}(f)= \left(M_{ii}M_{jj}\right)^{1/2} \mathrm{S}_{ij}(f)$ as the unnormalized cross-spectral density which is the Fourier transform of the mutual coherence function $<\phi_i(t+\tau)\phi_j^*(t)>$, the expectation value of the sample covariance can be expressed as

$$M_{ij} = \int_{-\infty}^{\infty} Q_{ij}(f)df. \tag{8}$$

For illustrative purposes, it is now assumed that the measurements across the array are cross-spectrally pure [1]. The mathematical expression of cross-spectral purity is $\gamma_{ij}(\tau)=v_{ij}\gamma(\tau)$, where $\gamma(\tau)=\gamma_{ii}(\tau)$, for all sensors *i* and *j.*

With these definitions, the variances of the real and imaginary parts of the sample covariance elements can be expressed as

$$\sigma^2_{R_{ij}(T)} = \frac{M_{ii}M_{jj}}{2\mu}\left(2|\mathrm{Re}(v_{ij})|^2 - |v_{ij}|^2 + 1\right), \tag{9a}$$

$$\sigma^2_{I_{ij}(T)} = \frac{M_{ii}M_{jj}}{2\mu}\left(2|\mathrm{Im}(v_{ij})|^2 - |v_{ij}|^2 + 1\right), \tag{9b}$$

where $R_{ij}(T)$ and $I_{ij}(T)$ are found to be uncorrelated. Here, μ is defined by

$$\mu = \left[\frac{1}{T}\int_{-\infty}^{\infty} \Delta\left(\frac{\tau}{T}\right)|\gamma(\tau)|^2 d\tau\right]^{-1}, \tag{10}$$

and the triangle function is defined as

$$\Delta(\tau)=1-|\tau| \qquad \text{for } |\tau|\leq 1, \tag{11a}$$

$$\Delta(\tau)=0 \qquad \text{elsewhere.} \tag{11b}$$

In terms of the spectral density S(f), which is the Fourier Transform of $\gamma(\tau)$, a useful spectral representation for μ is given by

$$\mu = \left[\frac{1}{T}\iint_{-\infty}^{\infty} S(f)S^*(f')\left(\frac{\sin(\pi T(f-f'))}{\pi(f-f')}\right)^2 df'df\right]^{-1}. \tag{12}$$

By appropriate substitution of the means and variances given above, Eq (5) yields the desired SNR for the continuous measurement case

$$\mathrm{SNR}\{S_{ij}(T)\}=\mu|v_{ij}|^2. \tag{13}$$

Therefore, the number of independent samples available in continuous measurement time T is μ, where μ need not be discrete but must be greater than or equal to one, as is evident by inspection of Eqs (10) and (12). It is important to realize that if the assumption of cross-spectral purity cannot be made, the number of independent samples $\mathrm{SNR}\{S_{ii}(T)\}$ would not necessarily be identical across all sensors i of the array, but would be given by Eqs (10) or (12) with $\gamma(\tau)$ and $\mathrm{S}(f)$ replaced by $\gamma_{ii}(\tau)$ and $\mathrm{S}_{ii}(f)$ respectively.

The continuous sample size μ can also be interpreted as the time-bandwidth product of the cross-spectrally pure field received by the array [1][2]. For example, the coherence time scale

$$\tau_c = \int_{-\infty}^{\infty} |\gamma(\tau)|^2 d\tau = \int_{-\infty}^{\infty} |\mathrm{S}(f)|^2 df, \tag{14}$$

measures the the inverse bandwidth of field fluctuations over an infinite time window.

It is useful to now consider the influence that the SNR of $S_{ij}(T)$ has on estimators found in ocean-acoustic interferometry, since these rely upon inter-sensor coherence. For such estimators, Eq (13) implies that as the intersensor measurements become less correlated under decreasing $|v_{ij}|$, longer averaging times are necessary to achieve the same resolution for a given parameter. For practical considerations, the temporal augmentation necessary for an off-diagonal element at i,j to attain the SNR that a diagonal element achieved in time T is expected to be $(T/|v_{ij}|^2-T)$.

4. Finite-time-averaged joint mutual-intensity statistics

The probability distribution for the complex sample covariance matrix $\mathbf{S}[N]$, which is the same as the joint distribution of all the mutual intensity measurements $S_{ij}[N]$,

$$P(\mathbf{S}) = N|N\mathbf{S}|^{N-N_\Phi} I^{-1}(\mathbf{M}) \exp\left\{-N \operatorname{tr}\left(\mathbf{M}^{-1}\mathbf{S}\right)\right\}, \tag{15}$$

is obtained after slight modification of the complex Wishart distribution [3] where

$$I(\mathbf{M}) = \pi^{\frac{N_\Phi}{2}(N_\Phi - 1)} \Gamma(N) \cdots \Gamma(N - N_\Phi + 1) |\mathbf{M}|^N . \tag{16}$$

When the sample size N is replaced by the time-bandwidth product μ, for a finite-time measurement of the cross-spectrally pure noise field defined by Eqs (10) and (12), Eq (15) becomes the joint probability distribution for the finite-time-averaged mutual intensities of a hydrophone array. This distribution is exact when the complex noise data is digitally sampled at the rate μ/T, so that there are μ independent samples during period T.

5. Finite-time-averaged individual mutual-intensity statistics

In terms of its complex magnitude α_{ij} and phase ψ_{ij}, the sample covariance element $S_{ij}[N]$ is distributed according to [4][1]

$$P_{\alpha_{ij},\psi_{ij}}(\alpha,\psi) = \frac{2N^{N+1}\alpha^N \exp\{[2N\alpha\rho_{ij}\cos(\psi - \theta_{ij})] / (h_{ij}(1-\rho_{ij}^2))\}}{\pi\Gamma(N)(1-\rho_{ij}^2)h_{ij}^{N+1}}$$
$$\times K_{N-1}\left(\frac{2N\alpha / h_{ij}}{1-\rho_{ij}^2}\right) \tag{17}$$

where ρ_{ij} is the magnitude and θ_{ij} is the phase of the complex coherence factor v_{ij}, and $h_{ij}=(M_{ii}M_{jj})^{1/2}$. The magnitude α_{ij} obeys the distribution [4][1]

$$P_{\alpha_{ij}}(\alpha) = \frac{4N^{N+1}\alpha^N}{\Gamma(N)(1-\rho_{ij}^2)h_{ij}^{N+1}} I_0\left(\frac{2\rho_{ij}N\alpha / h_{ij}}{1-\rho_{ij}^2}\right) K_{N-1}\left(\frac{2N\alpha / h_{ij}}{1-\rho_{ij}^2}\right), \tag{18}$$

which has familiar behavior as the sensors become perfectly correlated, where $\rho_{ij}=1$, and uncorrelated, where $\rho_{ij}=0$. Finally, the phase ψ_{ij} is distributed according to [4][1]

$$P_{\psi_{ij}}(\psi)=\frac{\Gamma(N+1/2)(1-\rho_{ij}^2)^N\rho_{ij}\cos(\psi-\theta_{ij})}{2\sqrt{\pi}\Gamma(N)(1-(\rho_{ij}\cos(\psi-\theta_{ij}))^2)^{N+1/2}} + \frac{(1-\rho_{ij}^2)^N}{2\pi}F[N,1;1/2;(\rho_{ij}\cos(\psi-\theta_{ij}))^2], \tag{19}$$

where $-\pi<\psi_{ij}\le\pi$.

When the sample size N is replaced by the time-bandwidth product μ, for a finite-time measurement of the cross-spectrally pure noise field defined by Eqs (10) and (12), Eqs (17-19) define the probability distributions for an individual finite-time-averaged mutual intensity measurement. Again, these distributions are exact when the complex noise data is digitally sampled at the rate μ/T.

6. Parameter resolution bounds

The N_g-dimensional parameter vector $\boldsymbol{g}$ can be inferred from the interferometric statistics **S**(N) when the expected value of the complex sample covariance is a function of $\boldsymbol{g}$, such that <**S**(N)>=**M**($\boldsymbol{g}$). For simplicity, let the matrix **M** be equivalently represented as a N_Φ^2-dimensional vector by stacking its columns. Let the inverse matrix $\mathbf{M}^{-1}$ also be taken as a vector by the same procedure. Since the complex sample covariance **S**(N) is a sufficient statistic for the estimation of **M**($\boldsymbol{g}$), any unbiased estimate $\hat{g}$ of parameter vector $\boldsymbol{g}$ given the statistics **S**(N), or equivalently the noise field data $\boldsymbol{\Phi}[1], \boldsymbol{\Phi}[2], \boldsymbol{\Phi}[3]\ldots \boldsymbol{\Phi}[N]$, is bounded by the error covariance

$$E[(\hat{\boldsymbol{g}}-\boldsymbol{g})(\hat{\boldsymbol{g}}-\boldsymbol{g})^{\mathrm{T}}]\ge\frac{1}{N}\left(\frac{\partial\mathbf{M}}{\partial\boldsymbol{g}}^{\mathrm{T}}(\mathbf{M}^{-1})(\mathbf{M}^{-1})^{\mathrm{T}}\frac{\partial\mathbf{M}}{\partial\boldsymbol{g}}\right)^{-1}, \tag{20}$$

where the right hand side is the inverse of the Fisher information matrix. It is noteworthy that while the matrix $\frac{\partial\mathbf{M}}{\partial\boldsymbol{g}}$ is not invertible in general, the bound may always be written in the form

$$E[(\hat{\boldsymbol{g}}-\boldsymbol{g})(\hat{\boldsymbol{g}}-\boldsymbol{g})^{\mathrm{T}}] \geq \frac{1}{N}\left(\frac{\partial \boldsymbol{g}}{\partial \mathbf{M}} \mathbf{M}\mathbf{M}^{\mathrm{T}} \frac{\partial \boldsymbol{g}}{\partial \mathbf{M}}^{\mathrm{T}}\right), \tag{21}$$

if the matrix $\frac{\partial \boldsymbol{g}}{\partial \mathbf{M}}$ exists [1] as it must when the problem is properly constrained. This form requires no matrix inversion, but it may be less plausible to implement than Eq (20) because the matrix $\frac{\partial \boldsymbol{g}}{\partial \mathbf{M}}$ is usually more difficult to determine than $\frac{\partial \mathbf{M}}{\partial \boldsymbol{g}}$ for applications in ocean-acoustic interferometry.

7. Discussion

While the results of this analysis are important from a general scientific perspective in that they broaden our understanding of the statistical laws that govern many underwater ambient noise measurements, they also have important consequences for practical issues in the analysis of ocean-acoustic ambient noise data.

For example, parameters such as as the geo-acoustic properties and geometry of the waveguide as well as the noise source characteristics affect not only the level but also the horizontal and vertical directional spectra of the noise. But these directional spectra must be measured by a form of acoustic interferometry that is statistical in nature due to the randomness of the noise. A quantitative analysis of the bias and resolution of a particular interferometric parameter estimate must therefore come from the mutual-intensity statistics presented here.

In a particular extension of the work presented here, it has recently been shown that the optimal method for finding either the expected spectral, temporal, spatial or angular dependence of ambient noise is to first take the log-transform of the respective noise intensity series and then look for the pattern by matched filtering with expected log-transformed patterns [5][2]. This has significant implications in fitting spectral trends, in correlating long-term noise time series with environmental stresses, and in determining the directional dependence of an ambient noise field.

References

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