Mean and covariance of the forward field propagated through a stratified ocean waveguide with three-dimensional random inhomogeneities

Purnima Ratilal and Nicholas C. Makris
Department of Ocean Engineering, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

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Compact analytic expressions are derived for the mean, mutual intensity, and spatial covariance of the acoustic field forward propagated though a stratified ocean waveguide containing three-dimensional random surface and volume inhomogeneities. The inhomogeneities need not obey a stationary random process in space, can be of arbitrary composition and size relative to the wavelength, or can have large surface roughness and slope. The form of the mean forward field after multiple scattering through the random waveguide is similar to that of the incident field, except for a complex change in the horizontal wave number of each mode. This change describes attenuation and dispersion induced by the medium's inhomogeneities, including potential mode coupling along the propagation path. The spatial covariance of the forward field between two receivers includes the accumulated effects of both coherent and incoherent multiple forward scattering through the random waveguide. It is expressed as a sum of modal covariance terms. Each term depends on the medium's expected modal extinction densities as well as the covariance of its scattering properties, which potentially couple each mode to every other mode. Three-dimensional scattering effects can become important at ranges where the Fresnel width exceeds the cross-range coherence scale of the medium's inhomogeneities. © *2005 Acoustical Society of America.* [DOI: 10.1121/1.1993087]

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I. INTRODUCTION

Compact analytic expressions are derived for the mean, mutual intensity, and spatial covariance of the acoustic field forward propagated through a stratified ocean waveguide with three-dimensional (3-D) random surface and volume inhomogeneities. The novelty and advantages of the approach stem from the fact that 3-D multiple forward scattering is included in a formulation where received field moments are analytically expressed in terms of the moments of the random medium's spatially varying scatter function[1,2] density. This makes it possible to describe volume and surface scatterers of arbitrary composition and size relative to the wavelength that individually may strongly scatter the acoustic field. This differs substantially from restrictive perturbation theory and Rayleigh[3]-Born[4] approximation methods, where parameters such as surface roughness, slope, or changes in medium properties must be small. It also differs from parabolic equation approaches, where field moments must be obtained by numerical marching algorithms. Since the inhomogeneities need not obey a stationary random process in space, the formulation accounts for expected range, cross-range, and depth-dependent variations in the medium's scatter function density. It is based upon a modal formulation for coherent 3-D scattering in an ocean waveguide[1,5] and the waveguide extinction or generalized forward scatter theorem,[6] both of which stem directly from Green's theorem.

After describing the approach in the context of previous work, an analytic expression is derived for the acoustic field forward scattered from a single elemental shell of random inhomogeneities between a point source and distant receiver in an ocean waveguide. This is done in Sec. II via a single-scatter approximation. Difference and integral equations are then developed to march the mean field and expected power through a randomly inhomogeneous waveguide in Secs. III and IV. This includes the accumulated effects of multiple forward scattering through the medium for both the *mean and covariance* of the forward field.

Compact solutions for the mean, variance, and second moment of the forward propagated field are given in terms of parameters necessary to describe the incident field as well as the mean and spatial covariance of the medium's scatter function density in Sec. V. Compact solutions for transmitted power and the signal-to-noise ratio of the forward field are also presented in Sec. V. They are used to analytically show that 3-D scattering effects can become important at ranges between the source and receiver, where the Fresnel width approaches and exceeds the cross-range coherence scale of the medium's inhomogeneities.

The approach is generalized to determine the mutual intensity and spatial covariance of the forward propagated field in Sec. VI. Compact solutions are given in Sec. VI B. These can be used to the quantify the effect of randomness in the ocean medium on ocean acoustic remote sensing with receiving arrays of arbitrary configuration.

The fundamental assumptions of the present paper are that (1) in the absence of inhomogeneities, the waveguide is horizontally stratified; (2) the medium can be described by a scattering process where single scattering is valid in horizontal range increments large enough for the modal diagonalization condition of Eq. (57) to be valid; (3) the medium's 3-D inhomogeneities obey a stationary random process within the incremental range scales described in (2) and the local Fresnel width in cross-range, but may not be stationary in the vertical or across larger horizontal scales; (4) the forward scattered field is approximated as that which successively forward scatterers through the inhomogeneous medium by interactions within the Fresnel width in cross-range from source to receiver; (5) the field scattered solely from inhomogeneities within any *single* horizontal range increment described in (2) must be small compared to the incident field; (6) to obtain mutual intensity from the power equations, waveguide modes are assumed to be uncorrelated, as is consistent with the central limit theorem, which is supported by many observations[7] and statistical theories for ocean-acoustic fluctuations.[8,9]

A. Approach in the context of previous developments

Some of the basic ideas behind the approach were inspired by the intuitive and physically compelling work of Rayleigh[10] in his explanation for the blue sky and red sunset, both of which have analogies in acoustic reverberation and transmission through the ocean. To study forward propagation in free space with random inhomogeneities, or the red sunset problem, Rayleigh first analyzed the effect of scattering by a thin slab of inhomogeneities in what we now call the *mean* forward field. He used this result to show that multiple forward scattering by inhomogeneities leads to an altered wave number in the mean field, which can be interpreted in terms of an effective medium when the scattered field from any slab is small compared to the incident field. The expected wave number change, ν, is directly proportional to the expected number of inhomogeneities per unit volume of the medium, n_V, multiplied by the forward scatter function of the inhomogeneities, S_f, and $2\pi/k^2$, where k is the wave number in free space; $\nu=(2\pi/k^2)\langle n_V S_f\rangle$. This wave number change leads to dispersion and attenuation in the mean forward field, and causes temporal distortion of the original signal waveform. A more modern version of Rayleigh's approach is given by van de Hulst,[11] where statistical operations leading to a mean field are again only implicit. The same results were later obtained by Foldy[12] using a significantly different multiple scatter series formulation, akin to Dyson's later more general formulation,[13] with ensemble averaging concepts but essentially the same physical assumptions as Rayleigh. Rayleigh implicitly, and van de Hulst explicitly, used a stationary phase approximation[14] to arrive at what we refer to here as the mean *direct wave*, following Rayleigh's intuitive terminology. The direct wave is comprised of the incident field and multiply scattered field contributions propagating within the Fresnel angle of forward from any inhomogeneous slab to the receiver. The latter are those contributions that travel with and can coherently interfere with the incident field. We refer to scattered field contributions that involve longer propagation paths, where some or all multiple scattering falls outside the Fresnel angle, to be *coda*. Coda is equivalent to *reverberation*. Many classic continuum models for optical and laser beam propagation through the turbulent atmosphere are based on assumptions similar to those in Rayleigh's original slab and direct wave formulation of forward scatter.[15]

The signal measured at a receiver from a point source in a waveguide is typically composed of multiple arrivals due to multimodal propagation effects, even when the medium is not random. We consider the *direct wave* in an inhomogeneous waveguide to be comprised of the incident and multiply scattered field contributions propagating within the Fresnel *azimuth* of forward from any cylindrical shell of inhomogeneities about the source to the receiver. Arrivals involving wider azimuthal scattering angles tend to form coda. The direct wave typically contains the most important information for signal transmission through the medium. Coda is important primarily in reverberation studies since it typically falls below the dynamic range or noise level of measurement systems designed for direct wave reception.

In a shallow-water waveguide, the effect of scattering by a single object leads to mode coupling and a redistribution of mode amplitudes that can be expressed in terms of the scatter function of the object.[1,5] We show that a distribution of volume or surface inhomogeneities over a sufficiently large range or depth increment causes the waveguide modes to decouple in the *mean* forward field under widely satisfied conditions. This allows the *mean* forward field to be analytically *marched* through range to include multiple forward scattering via a single modal sum by an explicit procedure analogous to Rayleigh's implicit marching procedure in free space. The resulting mean field has a form similar to the incident field, but with a change in the complex horizontal wave number for each mode due to scattering that leads to additional attenuation and dispersion. This wave number change is determined by the medium's expected scatter function density, which may vary as a function of range, depth, and azimuth. By invoking the generalized waveguide extinction theorem,[6] we show that attenuation from forward scattering for each mode can be expressed in terms of the expected waveguide modal extinction density of the medium. This provides a convenient method for estimating power losses after multiple forward scattering, given knowledge of the intrinsic scattering properties of the random medium.[16] We then apply a similar marching approach to derive analytic expressions for the mutual intensity and covariance of the forward field between two receivers in a random waveguide from a distant point source. We show that the resulting covariance can be expressed as a sum of modal covariance terms. Each term depends on the medium's expected modal extinction density as well as the covariance of the medium's scatter function density, which potentially couples each mode to all other modes.

While our method for obtaining the second moments of the forward field requires explicit statistical operations, it is still consistent with the implicit first moment analysis in Rayleigh's original explanation of the red sunset. To see this, let us follow Rayleigh's[10] notation in this Introduction and take m to be the dimensionless ratio of the scattered field to the incident field from a single slab of random inhomogeneities with surface normal in the direction of forward propagation. Rayleigh makes no distinction between m and its expectation $\langle m\rangle$ in his analysis, although his analysis is undoubtedly for the mean field. Let us assume then that m is a random variable, so that by m, Rayleigh is referring to what we here call $\langle m \rangle$. Rayleigh finds that $\langle m \rangle$ must be small but not negligible compared to unity in order to march the field and obtain his effective medium result. In our second moment analysis, we follow a similar asymptotic approach and find that if $\langle m \rangle$ is small, $\langle m \rangle^2$ must be negligible, since it is totally determined by $\langle m \rangle$. The second moment $\langle m^2 \rangle$, however, is not negligible since it contains the variance of m that need not depend on $\langle m \rangle$. So, we take $\langle m^2 \rangle$ to be small but not negligible compared to unity. This enables us to obtain the forward field's second moment by an analytic marching procedure similar to that used to obtain the mean. As Foldy has noted, the second moment of the field is often more important than the first in many experimental scenarios.[12]

Since Rayleigh, there has been a long history of development in the study of wave propagation through random media, especially in free space for which a number of excellent references exist, including those by Strobehn,[15] Tsang, Kong, and Shin,[17] Ishimaru,[18] and van de Hulst.[11] Rayleigh's effective medium approach, however, is still one of the most widely used since it conveniently expresses effective wave number changes in terms of the scatter function of medium inhomogeneities. For the ocean, analytic approaches using ray theory have been investigated for deep water applications by a number of authors.[19,20] More recently, ray and other approaches for the ocean, such as the 2-D parabolic equation, are now widely used to generate statistical realizations of the acoustic field.[21–24] In shallow water it has also become increasingly common to determine field moments via Monte Carlo simulations using the 2-D parabolic equation.[25]

Even with the current availability of computing power, analytic methods and solutions still offer advantages because they provide insight into the mechanisms that lead to the observed phenomena. This is one of the primary motivations for the present paper. Another motivation is that it is still very difficult to perform 3-D Monte-Carlo simulations of propagation through a random ocean waveguide. The present analytic technique is then also valuable because it can be employed to efficiently solve a wide range of practical 3-D problems. A modal formulation was selected because modes are the most clearly identifiable entities that may propagate with statistical independence in a random waveguide.

A number of modal formulations for propagation through a waveguide with randomness have been previously developed. To our knowledge, however, they are all based upon perturbation theory and either require surface roughness and slope to be small or variations in medium sound speed and density to be small. For mean field propagation, Bass, Freulicher, and Fuks[26,27] investigated the specific problem of propagation through a waveguide with small boundary roughness. They expressed the mean field in terms of complex modal wave number changes by solving Dyson's equation with perturbation theory.[26] Kuperman and Ingenito[28] later used Bass, Freulicher, and Fuks[26] results to investigate the attenuation of the mean sound field propagating through a shallow water waveguide with rough boundaries, following experimental work by Ingenito.[29] Their approach, however, neglects the effect of dispersion in the mean field. Kuperman and Schmidt[30,31] extended Kuperman and Ingenito's[28] 3-D perturbation theory approach for mean field attenuation to a more generalized wave number formulation for small amplitude and slope roughness at multiple layers. Tracey and Schmidt[32] used this to develop a 2-D modal formulation to describe mean field attenuation from small perturbations in medium sound speed and density or surface roughness and slope. They later extended this to describe the effect of small 3-D perturbations in seabed sound speed and density on mean field modal attenuation for inhomogeneities with vertical scales small compared to the acoustic wavelength.[33]

A number of purely single scatter approaches for estimating the second moment of the received field in a waveguide have been developed, including those by Sutton and McCoy[34] and Tracey and Schmidt.[32] While these second moment approaches include a single scatter contribution from each inhomogeneity, they ignore the multiple scattering effects between inhomogeneities along the forward propagation path. This limits their applicability to relatively short ranges, as noted by Sutton and McCoy.[34]

A number of formulations for second moment propagation in a waveguide that include multiple scattering between medium inhomogeneities along the propagation path have been developed. Bass, Freulicher, and Fuks[27] again investigated a waveguide with small boundary roughness, but this time started with the general Bethe-Salpeter equations.[35] They obtained approximate transfer equations for modal intensities, but no explicit solution for the field moments. Dozier and Tappert,[36] and later Creamer,[37] derived expressions for the second moment of the forward field in a 2-D ocean waveguide with small sound speed perturbations by following the perturbed coupled-mode approach of Marcuse[38] in fiber optics. Penland[39] addressed 3-D effects neglected by Dozier and Tappert[36] for small sound speed fluctuations that have slow range variation with respect to the acoustic wavelength by making a number of adiabatic assumptions, as noted by Frankenthal and Beran.[40] Continuing the small sound speed perturbation approximation and adopting a slab formulation, Frankenthal and Beran[40] derived difference equations for the mean and second moment of the field forward propagated through a 3-D random channel with a rigid bottom and a pressure-release top, but apparently did not solve these equations. They found, as do we, that energy is not eventually equipartioned among the modes, as has been suggested based on analysis of 2-D models such as those of Dozier and Tappert.[36] They also found that Penland's[39] approximations are inconsistent with their equation for the propagation of modal power.[40]

Here we present a formulation for 3-D propagation through a stratified waveguide with random surface or volume inhomogenieties of arbitrary size relative to the acoustic wavelength and arbitrary contrast in compressibility and density from the pure medium. Since the inhomogenieties may strongly scatter the incident field individually, this formulation is more general than approaches based on small perturbations in sound speed and density or surface roughness and slope. Our formulation also describes the accumulated effects of *multiple forward scattering* on the *mean and cova-*

riance of the forward propagated field. These include coherent, partially coherent, and incoherent interactions with the incident field that lead to attenuation, dispersion, and exponential coefficients of field variance that describe mode coupling induced by the medium's inhomogeneities. We obtain difference equations, integral equations, and *analytic solutions* to these equations for the mean field, the expected intensity and the mutual intensity, including evanscent effects. We find in a following paper[16] that the present 3-D formulation is consistent with standard 2-D Monte-Carlo simulations at ranges within which the 2-D simulations should be valid, i.e., where the Fresnel angle of the incident field is less than the angle spanned by the cross-range extent of the inhomogeneity. This follows the analytic predictions of the present paper. This work has been presented at numerous meetings of the Acoustical Society of America,[41,42] including applications involving random internal waves,[43–46] random seabed inhomogeneities,[43,44,47] random bubble clouds,[43,48] and source localization in a random ocean waveguide.[49]

II. SCATTERED FIELD CONTRIBUTION TO THE DIRECT WAVE FROM A SHELL OF INHOMOGENEITIES

An analytic expression is derived for the scattered field contribution to the direct wave from an elemental cylindrical shell of inhomogeneities between the source and receiver. This is the portion of the field scattered from the shell that coherently interferes with the incident field, on average.

The origin of the coordinate system is placed at the air-water interface with the positive z axis pointing downward. The source is located at the horizontal origin $\mathbf{r_0}=(0,0,z_0)$, while receiver coordinates are given by $\mathbf{r}=(x,y,z)$ and those of inhomogeneity centers by $\mathbf{r_t}=(x_t,y_t,z_t)$. Spatial cylindrical (ρ,ϕ,z) and spherical systems (r,θ,ϕ) are defined by $x=r\sin\theta\cos\phi$, $y=r\sin\theta\sin\phi$, $z=r\cos\theta$, and $\rho^2=x^2+y^2$. The horizontal and vertical wave number components for the nth mode are, respectively, $\xi_n=k\sin\alpha_n$ and $\gamma_n=k\cos\alpha_n$, where α_n is the elevation angle of the mode measured from the z axis. Here, $0\leqslant\alpha_n\leqslant\pi/2$ so that the down and upgoing plane wave components of each mode have elevation angles α_n and $\pi-\alpha_n$, respectively. The corresponding vertical wave number of the down- and upgoing components of the nth mode are γ_n and $-\gamma_n$, respectively, where $\Re\{\gamma_n\}\geqslant 0$. The azimuth angle of the mode is denoted by β. The wave number magnitude k equals the angular frequency ω divided by the sound speed c in the object layer, where $k^2=\xi_n^2+\gamma_n^2$. The geometry of spatial and wave number coordinates is shown in Ref. 50.

The field scattered to $\mathbf{r}$ from inhomogeneities within a cylindrical shell of radius ρ_s and thickness $\Delta\rho_s$ is found by integrating volumetric contributions with a single scatter approximation,

$$\Phi_s(\mathbf{r}|\mathbf{r_0},\Delta\rho_s(\rho_s))=\iiint_{\Delta V_s}\varphi_s(\mathbf{r}|\mathbf{r_0},\mathbf{r_t})\mathrm{d}V_t, \tag{1}$$

where ΔV_s is the volume of the cylindrical shell element and $\varphi_s(\mathbf{r}|\mathbf{r_0},\mathbf{r_t})$ is the scattered field per unit volume from inhomogeneities *centered* at $\mathbf{r_t}$.

A modal solution for the 3-D bistatic scattered field from an inhomogeneity of arbitrary size, shape, and material properties in an ocean waveguide has been derived from Green's theorem in Refs. 1, 5, and 51. With this, the scattered field per unit volume can be written as

$$\begin{aligned}\varphi_s(\mathbf{r}|\mathbf{r_0},\mathbf{r_t})&\\ =\sum_{m=1}^{M_{\max}}\sum_{n=1}^{M_{\max}}\frac{(4\pi)^2}{k}&\\ \times[A_m(\mathbf{r}-\mathbf{r_t})A_n(\mathbf{r_t}-\mathbf{r}_0)s_{\mathbf{r_t}}(\pi-\alpha_m,\beta_s(\phi,\phi_t);\alpha_n,\phi_t)&\\ -B_m(\mathbf{r}-\mathbf{r_t})A_n(\mathbf{r_t}-\mathbf{r}_0)s_{\mathbf{r_t}}(\alpha_m,\beta_s(\phi,\phi_t);\alpha_n,\phi_t)&\\ -A_m(\mathbf{r}-\mathbf{r_t})B_n(\mathbf{r_t}-\mathbf{r}_0)s_{\mathbf{r_t}}(\pi-\alpha_m,\beta_s(\phi,\phi_t);\pi-\alpha_n,\phi_t)&\\ +B_m(\mathbf{r}-\mathbf{r_t})B_n(\mathbf{r_t}-\mathbf{r}_0)s_{\mathbf{r_t}}(\alpha_m,\beta_s(\phi,\phi_t);\pi-\alpha_n,\phi_t)],&\end{aligned} \tag{2}$$

where $A_n(\mathbf{r_t}-\mathbf{r}_0)$ and $B_n(\mathbf{r_t}-\mathbf{r}_0)$ are the amplitudes of the *down-* and *upgoing* modal plane wave components incident on the inhomogeneity at $\mathbf{r_t}$, $A_m(\mathbf{r}-\mathbf{r_t})$ and $B_m(\mathbf{r}-\mathbf{r_t})$ are the amplitudes of the *up-* and *downgoing* modal components scattered from the inhomogeneity,[51] $s_{\mathbf{r_t}}(\alpha,\beta;\alpha_i,\beta_i)$ is the scatter function[2] density of the medium at $\mathbf{r_t}$, $\beta_s(\phi,\phi_t)=\phi-\arcsin\{(\rho_t/|\boldsymbol{\rho}-\boldsymbol{\rho}_\mathbf{t}|)\sin(\phi_t-\phi)\}$ is the azimuth of the receiver from the target, α_n are the previously defined modal elevation angles, and $M_{\max}$ is the mode number at which the modal summations can be truncated and still accurately represent the field. We abbreviate the modal summation notation in subsequent equations with the understanding that the sum is taken to $M_{\max}$ modes.

The scatter function density describes scattering from both discrete and continuously varying inhomogeneities, as shown in Appendix A. The scattered field must be described in terms of a double modal sum to account for coupling of incident and scattered modes by the inhomogeneity. Equation (2) is applicable when the source and receiver ranges are sufficiently far from the target that the plane wave scatter function description is valid.[5,52,54]

For a single shell, the incident field has modal plane wave amplitudes given by,[1,5,51]

$$A_n(\mathbf{r_t}-\mathbf{r_0})=\frac{i}{d(z_0)}\frac{1}{\sqrt{8\pi\xi_n\rho_t}}u_n(z_0)N_n^{(1)}\mathrm{e}^{i(\xi_n\rho_t+\gamma_n z_t-\pi/4)}, \tag{3}$$

$$B_n(\mathbf{r_t}-\mathbf{r_0})=\frac{i}{d(z_0)}\frac{1}{\sqrt{8\pi\xi_n\rho_t}}u_n(z_0)N_n^{(2)}\mathrm{e}^{i(\xi_n\rho_t-\gamma_n z_t-\pi/4)}, \tag{4}$$

while the scattered modal plane wave amplitudes in Eq. (2) are[1,5,51]

$$A_m(\mathbf{r}-\mathbf{r_t})=\frac{i}{d(z_t)}\frac{1}{\sqrt{8\pi\xi_m|\boldsymbol{\rho}-\boldsymbol{\rho}_\mathbf{t}|}}u_m(z)N_m^{(1)}\mathrm{e}^{i(\xi_m|\boldsymbol{\rho}-\boldsymbol{\rho}_\mathbf{t}|+\gamma_m z_t-\pi/4)}, \tag{5}$$

$$B_m(\mathbf{r}-\mathbf{r_t}) = \frac{i}{d(z_t)}\frac{1}{\sqrt{8\pi\xi_m|\boldsymbol{\rho}-\boldsymbol{\rho}_\mathbf{t}|}}u_m(z)N_m^{(2)}\mathrm{e}^{i(\xi_m|\boldsymbol{\rho}-\boldsymbol{\rho}_\mathbf{t}|-\gamma_m z_t-\pi/4)}. \tag{6}$$

Upon substituting Eqs. (3), (4), (5), and (6) into Eq. (2), and Eq. (2) into Eq. (1), the scattered field from the shell becomes

$$\begin{aligned}
&\Phi_s(\mathbf{r}|\mathbf{r_0},\Delta\rho_s(\rho_s)) \\
&\quad = \sum_m\sum_n \iiint_{\Delta V_s}\frac{2\pi}{k}\frac{i}{d(z_0)d(z_t)}\frac{1}{\sqrt{\xi_m\xi_n|\boldsymbol{\rho}-\boldsymbol{\rho}_\mathbf{t}|\rho_t}}u_m(z)u_n(z_0)\mathrm{e}^{i(\xi_m|\boldsymbol{\rho}-\boldsymbol{\rho}_\mathbf{t}|+\xi_n\rho_t)} \\
&\quad\quad\times[N_m^{(1)}N_n^{(1)}\mathrm{e}^{i(\gamma_m+\gamma_n)z_t}s_{\mathbf{r_t}}(\pi-\alpha_m,\beta_s(\phi,\phi_t);\alpha_n,\phi_t) \\
&\quad\quad-N_m^{(2)}N_n^{(1)}\mathrm{e}^{i(-\gamma_m+\gamma_n)z_t}s_{\mathbf{r_t}}(\alpha_m,\beta_s(\phi,\phi_t);\alpha_n,\phi_t) \\
&\quad\quad-N_m^{(1)}N_n^{(2)}\mathrm{e}^{i(\gamma_m-\gamma_n)z_t}s_{\mathbf{r_t}}(\pi-\alpha_m,\beta_s(\phi,\phi_t);\pi-\alpha_n,\phi_t) \\
&\quad\quad+N_m^{(2)}N_n^{(2)}\mathrm{e}^{i(-\gamma_m-\gamma_n)z_t}s_{\mathbf{r_t}}(\alpha_m,\beta_s(\phi,\phi_t);\pi-\alpha_n,\phi_t)]\rho_t\,\mathrm{d}\rho_t\,\mathrm{d}\phi_t\,\mathrm{d}z_t,
\end{aligned} \tag{7}$$

where $\xi_m|\boldsymbol{\rho}-\boldsymbol{\rho}_\mathbf{t}|=\xi_m\sqrt{\rho^2+\rho_t^2-2\rho\rho_t\cos(\phi_t-\phi)}$.

The integration over shell azimuth ϕ_t leads on average to two stationary phase contributions centered at $\phi_t=\phi$ and $\phi_t=\phi+\pi$ over angular widths $\phi_F(\rho,\rho_t)=\sqrt{2\pi(\rho-\rho_t)/\xi_m\rho\rho_t}$ and $\phi_B(\rho,\rho_t)=\sqrt{2\pi(\rho+\rho_t)/\xi_m\rho\rho_t}$, respectively, via approximations of the form

$$\begin{aligned}
&\int_0^{2\pi}\frac{1}{\sqrt{|\boldsymbol{\rho}-\boldsymbol{\rho}_\mathbf{t}|}}\mathrm{e}^{i\xi_m|\boldsymbol{\rho}-\boldsymbol{\rho}_\mathbf{t}|}s_{\mathbf{r_t}}(\pi-\alpha_m,\beta_s(\phi,\phi_t);\alpha_n,\phi_t)\mathrm{d}\phi_t \\
&\quad=\int_0^{2\pi}\frac{1}{[\rho^2+\rho_t^2-2\rho\rho_t\cos(\phi_t-\phi)]^{(1/4)}}\mathrm{e}^{i\xi_m\sqrt{\rho^2+\rho_t^2-2\rho\rho_t\cos(\phi_t-\phi)}}s_{\mathbf{r_t}}(\pi-\alpha_m,\beta_s(\phi,\phi_t);\alpha_n,\phi_t)\mathrm{d}\phi_t \\
&\quad\approx \mathrm{e}^{i\pi/4}\frac{\mathrm{e}^{i\xi_m(\rho-\rho_t)}}{\sqrt{\rho-\rho_t}}\int_{\phi-\phi_F/2}^{\phi+\phi_F/2}s_{\mathbf{r_t}}(\pi-\alpha_m,\beta_s(\phi,\phi_t);\alpha_n,\phi_t)\mathrm{e}^{i\xi_m[\rho\rho_t/2(\rho-\rho_t)](\phi_t-\phi)^2}\,\mathrm{d}\phi_t \\
&\quad\quad+\mathrm{e}^{-i\pi/4}\frac{\mathrm{e}^{i\xi_m(\rho+\rho_t)}}{\sqrt{\rho+\rho_t}}\int_{\phi+\pi-\phi_B/2}^{\phi+\pi+\phi_B/2}s_{\mathbf{r_t}}(\pi-\alpha_m,\beta_s(\phi,\phi_t);\alpha_n,\phi_t)\mathrm{e}^{i\xi_m[\rho\rho_t/2(\rho+\rho_t)](\phi_t-\phi-\pi)^2}\,\mathrm{d}\phi_t.
\end{aligned} \tag{8}$$

These define the active region from which dominant contributions to the scattered field are expected to arise. The first term describes forward scatter through the shell within the Fresnel angle ϕ_F, while the second describes backscatter from the rear of the shell. The backscatter term tends not to contribute significantly to the direct wave because it requires longer propagation paths and so tends to (1) suffer greater transmission loss and (2) not arrive with the direct wave in pulsed transmissions. The backscattered contributions also tend to cancel on average since they have phases that oscillate with shell thickness $\Delta\rho_s$.

By applying the type of integral approximations made in Eq. (8) to Eq. (7), the direct wave portion of the scattered field from the shell is found to be

$$\begin{aligned}
&\Phi_s(\mathbf{r}|\mathbf{r_0},\Delta\rho_s(\rho_s)) \\
&\quad=\sum_m\sum_n\int_0^\infty \mathrm{d}z_t\int_{\rho_s-\Delta\rho_s/2}^{\rho_s+\Delta\rho_s/2}\mathrm{d}\rho_t\int_{\phi-\phi_F/2}^{\phi+\phi_F/2}\rho_t\,\mathrm{d}\phi_t\,\mathrm{e}^{i\xi_m[\rho\rho_t/2(\rho-\rho_t)](\phi_t-\phi)^2} \\
&\quad\quad\times\frac{2\pi}{k}\frac{i}{d(z_0)d(z_t)}\frac{1}{\sqrt{\xi_m\xi_n(\rho-\rho_t)\rho_t}}u_m(z)u_n(z_0)\mathrm{e}^{i\pi/4}\mathrm{e}^{i\xi_m\rho}\mathrm{e}^{i(\xi_n-\xi_m)\rho_t} \\
&\quad\quad\times[N_m^{(1)}N_n^{(1)}\mathrm{e}^{i(\gamma_m+\gamma_n)z_t}s_{\mathbf{r_t}}(\pi-\alpha_m,\beta_s(\phi,\phi_t);\alpha_n,\phi_t) \\
&\quad\quad-N_m^{(2)}N_n^{(1)}\mathrm{e}^{i(-\gamma_m+\gamma_n)z_t}s_{\mathbf{r_t}}(\alpha_m,\beta_s(\phi,\phi_t);\alpha_n,\phi_t) \\
&\quad\quad-N_m^{(1)}N_n^{(2)}\mathrm{e}^{i(\gamma_m-\gamma_n)z_t}s_{\mathbf{r_t}}(\pi-\alpha_m,\beta_s(\phi,\phi_t);\pi-\alpha_n,\phi_t) \\
&\quad\quad+N_m^{(2)}N_n^{(2)}\mathrm{e}^{i(-\gamma_m-\gamma_n)z_t}s_{\mathbf{r_t}}(\alpha_m,\beta_s(\phi,\phi_t);\pi-\alpha_n,\phi_t)].
\end{aligned} \tag{9}$$

The active region for forward scattering on the shell then occurs over a Fresnel width $Y_F(\rho,\rho_s)=\rho_s\phi_F(\rho,\rho_s) =\sqrt{2\pi(\rho-\rho_s)\rho_s/\xi_m\rho}$, which is a function of the horizontal wave number of the mth mode. Long-range propagation is directed near the horizontal in most ocean waveguides, so that the mode-independent $Y_{\mathrm{F}}(\rho,\rho_s)\approx\sqrt{\lambda(\rho-\rho_s)\rho_s/\rho}$ is both a good and practical approximation. The Fresnel width varies symmetrically between the source and receiver as a consequence of reciprocity. The maximum Fresnel width $Y_{\mathrm{F}}(\rho,\rho/2)=(\rho/2)\phi_{\mathrm{F}}(\rho,\rho/2)=\sqrt{\lambda\rho/4}$ occurs at the midpoint between source and receiver and increases with the square root of their separation.

III. DIFFERENCE AND INTEGRAL EQUATIONS TO MARCH THE MEAN FIELD, POWER, AND EXPECTED INTENSITY THROUGH A WAVEGUIDE WITH RANDOM INHOMOGENEITIES

A. Mean forward field

An analytic expression is derived for the mean forward field propagated from a point source to a distant receiver through a stratified ocean waveguide containing random volume or surface inhomogeneities. This is done by first developing a difference equation that describes the change in the mean forward field at the receiver due to scattering from an elemental shell of inhomogeneities between the source and receiver. The mean field is then analytically marched through all shells to determine the effect of multiple forward scattering between the source and the receiver. The derivation of the mean scattered field from a single elemental shell and limiting conditions on its validity are given in Sec. IV.

It is first assumed that the inhomogeneities are confined within a cylinder of radius ρ_s centered at the source so that there are no inhomogeneities in the medium outside this cylinder. Let $\Phi(\mathbf{r}|\mathbf{r_0})$ be the direct wave measured at receiver $\mathbf{r}$ from the source at $\mathbf{r_0}$. The thickness of the cylinder containing inhomogeneities is now augmented by a small amount, $\Delta\rho_s$. Let $\Phi_s(\mathbf{r}|\mathbf{r_0},\Delta\rho_s(\rho_s))$ be the *scattered* field at the receiver from inhomogeneities in this new cylindrical shell of thickness $\Delta\rho_s$. Let the *total* field at the receiver given the new cylinder of radius $\rho_s+\Delta\rho_s$ be $\Phi(\mathbf{r}|\mathbf{r_0},\Delta\rho_s(\rho_s))$. Then the difference equation,

$$\Phi(\mathbf{r}|\mathbf{r_0},\Delta\rho_s(\rho_s)) = \Phi(\mathbf{r}|\mathbf{r_0}) + \Phi_s(\mathbf{r}|\mathbf{r_0},\Delta\rho_s(\rho_s)), \qquad (10)$$

is obtained where $\Phi_s(\mathbf{r}|\mathbf{r_0},\Delta\rho_s(\rho_s))$ is given by Eq. (9), as long as the width $\Delta\rho_s$ is sufficiently small for the single scatter approximation to be valid within it.

The mean field can be obtained by taking the expected value of Eq. (10),

$$\langle\Phi(\mathbf{r}|\mathbf{r_0},\Delta\rho_s(\rho_s))\rangle = \langle\Phi(\mathbf{r}|\mathbf{r_0})\rangle + \langle\Phi_s(\mathbf{r}|\mathbf{r_0},\Delta\rho_s(\rho_s))\rangle. \qquad (11)$$

The mean field in the absence of the shell can be expressed as a sum of normal modes,

$$\langle\Phi(\mathbf{r}|\mathbf{r_0})\rangle = \sum_n \langle\Phi^{(n)}(\mathbf{r}|\mathbf{r_0})\rangle, \qquad (12)$$

where $\langle\Phi^{(n)}(\mathbf{r}|\mathbf{r_0})\rangle$ is the contribution to the field by mode n. Equation (12) is generally valid even when the modes are coupled by scatterers in the waveguide since another modal summation can always be embedded within $\langle\Phi^{(n)}(\mathbf{r}|\mathbf{r_0})\rangle$.

The scattered field contribution to the direct wave was found using a stationary phase approximation in Sec. II. The mean of this scattered field is expressible as a single modal sum,

$$\langle\Phi_s(\mathbf{r}|\mathbf{r_0},\Delta\rho_s(\rho_s))\rangle = \sum_n \langle\Phi^{(n)}(\mathbf{r}|\mathbf{r_0})\rangle i\nu_n(\rho_s)\Delta\rho_s, \qquad (13)$$

under widely applicable conditions, as will be shown in Sec. IV A, where ν_n is the horizontal wave number change of the nth mode due to scattering by inhomogeneities in the medium. An analytic expression for this wave number change is given in Eq. (60). It depends on the expected scattering properties of the inhomogeneities in the *forward azimuth*.

Expressing the mean total field at the receiver as a modal sum,

$$\langle\Phi(\mathbf{r}|\mathbf{r_0},\Delta\rho_s(\rho_s))\rangle = \sum_n \langle\Phi^{(n)}(\mathbf{r}|\mathbf{r_0},\Delta\rho_s(\rho_s))\rangle \qquad (14)$$

and substituting Eqs. (12), (13), and (14) into Eq. (11), it follows that for each mode n,

$$\langle\Phi^{(n)}(\mathbf{r}|\mathbf{r_0},\Delta\rho_s(\rho_s))\rangle = \langle\Phi^{(n)}(\mathbf{r}|\mathbf{r_0})\rangle(1 + i\nu_n(\rho_s)\Delta\rho_s). \qquad (15)$$

This can be rewritten as the difference equation,

$$\Delta\langle\Phi^{(n)}(\mathbf{r}|\mathbf{r_0})\rangle = \langle\Phi^{(n)}(\mathbf{r}|\mathbf{r_0})\rangle i\nu_n(\rho_s)\Delta\rho_s, \qquad (16)$$

where $\Delta\langle\Phi^{(n)}(\mathbf{r}|\mathbf{r_0})\rangle=(\langle\Phi^{(n)}(\mathbf{r}|\mathbf{r_0},\Delta\rho_s(\rho_s))\rangle-\langle\Phi^{(n)}(\mathbf{r}|\mathbf{r_0})\rangle)$ describes the change in the nth mode's contribution to the mean field at the receiver as a result of scattering by inhomogeneities in the shell. From Eqs. (13) and (16), the modal sum of these changes equals the forward scattered field from the shell. Equation (16) can be recast as the integral equation,

$$\int_{\Psi_i^{(n)}}^{\langle\Psi_T^{(n)}\rangle} \frac{\mathrm{d}\langle\Phi^{(n)}(\mathbf{r}|\mathbf{r_0})\rangle}{\langle\Phi^{(n)}(\mathbf{r}|\mathbf{r_0})\rangle} = i\int_0^{\rho} \nu_n(\rho_s)\mathrm{d}\rho_s, \qquad (17)$$

which marches the mean forward field for each mode through the inhomogeneous medium since the inhomogeneities in adjacent single-scatter shells are assumed to be uncorrelated with each other. This includes multiple forward scattering from source to receiver in a manner analogous to that used by Rayleigh[10,11] and others[15,18] for free space.

In the absence of inhomogeneities in the medium, the field measured at the receiver for each mode is simply the incident field,[53]

$$\Psi_i^{(n)}(\mathbf{r}|\mathbf{r_0}) = 4\pi\frac{i}{d(z_0)\sqrt{8\pi}}\mathrm{e}^{-i\pi/4}u_n(z)u_n(z_0)\frac{\mathrm{e}^{i\xi_n\rho}}{\sqrt{\xi_n\rho}}, \qquad (18)$$

where $u_n(z)$ is the amplitude of the mode shape and $d(z)$ is the density at depth z. Within any approximately isovelocity layer of the waveguide, the mode shape can be expressed as

$$u_n(z) = N_n^{(1)}(z)\mathrm{e}^{i\gamma_n z} - N_n^{(2)}(z)\mathrm{e}^{-i\gamma_n z}, \qquad (19)$$

integrating Eq. (17), we have

$$\langle \Psi_T^{(n)}(\mathbf{r}|\mathbf{r_0})\rangle = \Psi_i^{(n)}(\mathbf{r}|\mathbf{r_0})\mathrm{e}^{i\int_0^\rho \nu_n(\rho_s)\mathrm{d}\rho_s}. \tag{20}$$

Summing the contribution from all the modes, the total mean forward field at the receiver in a waveguide containing inhomogeneities is then

$$\langle \Psi_T(\mathbf{r}|\mathbf{r_0})\rangle = \sum_n \langle \Psi_T^{(n)}(\mathbf{r}|\mathbf{r_0})\rangle, \tag{21}$$

$$= \sum_n \Psi_i^{(n)}(\mathbf{r}|\mathbf{r_0})\mathrm{e}^{i\int_0^\rho \nu_n(\rho_s)\mathrm{d}\rho_s}. \tag{22}$$

Here ν_n, given in Eq. (60), describes the change in the horizontal wave number of mode n as it propagates through the random inhomogeneous waveguide. It is a complex quantity that provides a measure of the attenuation and dispersion of each mode per unit horizontal range due to scattering by inhomogeneities in the medium. The real part of ν_n determines modal dispersion while the imaginary part of ν_n determines modal attenuation. If the source is at a null for a particular mode, that mode may still contribute to the field at a distant receiver through scattering. This is because the imaginary part of ν_n includes the effect of mode coupling along the propagation path through a sum over all modes by the waveguide extinction theorem.[6]

For the present formulation to be valid, the change in the mean forward field at the receiver due to scattering from any shell increment $\Delta\rho_s$ must be small relative to the incident field, as can be seen from Eqs. (15) and (22). Here $\Delta\rho_s$ must be small enough for the single scatter approximation to be valid within it, and large enough for the modal decoupling conditions derived in Sec. IV to be valid. Any individual inhomogeneity may have a large scattered field in its vicinity, on the order of the incident field, as is typical in shadow formation. In such cases, the present theory is still valid as long as the separation between inhomogeneities in the forward direction is large enough for the scattered field from the previous interaction to be small in comparison to the total incident field of the current interaction.

Both the difference equation (16) and integral equation (17) account for range-dependent variation in the medium's expected scatter function density. If the inhomogeneities are contained in an evanescent layer, such as in the sea bottom, Eq. (19) can still be used if the given mode is evanescent. In this case, we set $N_n^{(2)}=0$ and can write $N_n^{(1)}=u_n(H)\mathrm{e}^{\Im\{\gamma_n\}H}$, where H is the water depth, since the vertical wave number becomes purely imaginary.

B. Power of the forward field

Here we develop a difference equation that describes the change in the expected depth-integrated second moment of the forward field due to scattering from an elemental shell of inhomogeneities between the source and receiver. This is proportional to the corresponding change in power at a given range for a receiver of fixed area spanning the water column. The depth integrated second moment is then marched through the random waveguide to include multiple forward scattering between the source and receiver. The derivation of the expected power transmission through a single elemental shell of inhomogeneities is given in Sec. IV along with its validity conditions.

From Eq. (10), the second moment of the total field at receiver $\mathbf{r}$ from a source at $\mathbf{r_0}$ after scattering from a shell of inhomogeneities of thickness $\Delta\rho_s$ at range ρ_s can be expressed as

$$\begin{aligned}\langle |\Phi(\mathbf{r}|\mathbf{r_0},\Delta\rho_s(\rho_s))|^2\rangle &= \langle |\Phi(\mathbf{r}|\mathbf{r_0})|^2\rangle + \langle |\Phi_s(\mathbf{r}|\mathbf{r_0},\Delta\rho_s(\rho_s))|^2\rangle \\ &\quad + \langle \Phi(\mathbf{r}|\mathbf{r_0})\Phi_s^*(\mathbf{r}|\mathbf{r_0},\Delta\rho_s(\rho_s))\rangle \\ &\quad + \langle \Phi^*(\mathbf{r}|\mathbf{r_0})\Phi_s(\mathbf{r}|\mathbf{r_0},\Delta\rho_s(\rho_s))\rangle. \end{aligned}\tag{23}$$

The first term on the right-hand side of Eq. (23) is the second moment of the received field in the absence of the shell. Next is the second moment of the field scattered from the shell. This is followed by two cross-terms arising from coherent interaction between incident and scattered fields.

We next integrate Eq. (23) over receiver depth to obtain the depth-integrated second moment of the field at horizontal range ρ. Integrating the first term to the right of Eq. (23), leads to the depth-integrated second moment of the field in the absence of the shell. It can always be expressed as a single modal sum,

$$\int_0^\infty \frac{1}{d(z)}\langle |\Phi(\mathbf{r}|\mathbf{r_0})|^2\rangle \mathrm{d}z = \sum_n \langle W^{(n)}(\boldsymbol{\rho}|\mathbf{r_0})\rangle, \tag{24}$$

even when the modes are coupled due to scattering in the waveguide, since other modal summations can always be embedded within $W^{(n)}(\boldsymbol{\rho}|\mathbf{r_0})$.

The depth integral of the second moment of the scattered field from the shell is expressible as a single modal sum,

$$\int_0^\infty \frac{1}{d(z)}\langle |\Phi_s(\mathbf{r}|\mathbf{r_0},\Delta\rho_s(\rho_s))|^2\rangle \mathrm{d}z = \sum_n \langle W^{(n)}(\boldsymbol{\rho}|\mathbf{r_0})\rangle \mu_n(\rho_s)\Delta\rho_s, \tag{25}$$

under widely satisfied conditions, as shown in Sec. IV B. Here $\mu_n(\rho_s)$ is what we refer to as the *exponential coefficient of modal field variance*. It accounts for the covariance of the medium's scatter function density and the potential coupling of energy from each mode n to every other mode in the waveguide as a result of the random scattering process. It is expressible as a single modal sum, as shown in Sec. IV B.

Depth-integration of the last two terms on the right in Eq. (23) leads to

$$\begin{aligned}\int_0^\infty \frac{1}{d(z)}[\langle \Phi(\mathbf{r}|\mathbf{r_0})\Phi_s^*(\mathbf{r}|\mathbf{r_0},\Delta\rho_s(\rho_s))\rangle &+ \langle \Phi^*(\mathbf{r}|\mathbf{r_0})\Phi_s(\mathbf{r}|\mathbf{r_0},\Delta\rho_s(\rho_s))\rangle]\mathrm{d}z \\ &= -\sum_n \langle W^{(n)}(\boldsymbol{\rho}|\mathbf{r_0})\rangle 2\Im\{\nu_n(\rho_s)\}\Delta\rho_s,\end{aligned}\tag{26}$$

which depends on the attenuation coefficient $\Im\{\nu_n\}$, as shown in Sec. IV C. Equation (26) reflects the fact that it is the coherent interaction between the incident and forward scattered fields that leads to extinction of the total forward field.

Each term on the right-hand side of Eq. (23) is now expressed as a single modal sum after depth-integration throughout the waveguide. As before, the depth integral of the left-hand side of Eq. (23) can always be expressed as a single modal sum via

$$\int_0^\infty \frac{1}{d(z)}\langle \Phi(\mathbf{r}|\mathbf{r_0},\Delta\rho_s(\rho_s))\Phi^*(\mathbf{r}|\mathbf{r_0},\Delta\rho_s(\rho_s))\rangle \mathrm{d}z = \sum_n \langle W^{(n)}(\boldsymbol{\rho}|\mathbf{r_0},\Delta\rho_s(\rho_s))\rangle. \quad (27)$$

Substituting Eqs. (24), (25), (26), and (27) for the various terms obtained after integrating Eq. (23) over depth leads to

$$\langle W^{(n)}(\boldsymbol{\rho}|\mathbf{r_0},\Delta\rho_s(\rho_s))\rangle = \langle W^{(n)}(\boldsymbol{\rho}|\mathbf{r_0})\rangle[1+(\mu_n(\rho_s)-2\Im\{\nu_n(\rho_s)\})\Delta\rho_s]. \quad (28)$$

This can be rewritten as the difference equation,

$$\Delta\langle W^{(n)}(\boldsymbol{\rho}|\mathbf{r_0})\rangle = \langle W^{(n)}(\boldsymbol{\rho}|\mathbf{r_0})\rangle(\mu_n(\rho_s)-2\Im\{\nu_n(\rho_s)\})\Delta\rho_s, \quad (29)$$

where $\Delta\langle W^{(n)}(\boldsymbol{\rho}|\mathbf{r_0})\rangle = (\langle W^{(n)}(\boldsymbol{\rho}|\mathbf{r_0},\Delta\rho_s(\rho_s))\rangle - \langle W^{(n)}(\boldsymbol{\rho}|\mathbf{r_0})\rangle)$ is the change in the depth-integrated second moment of the forward field due to scattering from inhomogeneities in the shell. This change has a component that depends on the interference between the incident and forward scattered fields.

Equation (29) can be recast as an integral equation,

$$\int_{W_i^{(n)}}^{\langle W_T^{(n)}\rangle} \frac{\mathrm{d}\langle W^{(n)}(\boldsymbol{\rho}|\mathbf{r_0})\rangle}{\langle W^{(n)}(\boldsymbol{\rho}|\mathbf{r_0})\rangle} = \int_0^\rho (\mu_n(\rho_s)-2\Im\{\nu_n(\rho_s)\})\mathrm{d}\rho_s, \quad (30)$$

which marches the depth-integrated second moment of the forward field for each mode through the inhomogeneous medium to include multiple forward scattering along the way. This approach is based on the assumption that inhomogeneities in adjacent single-scatter shells are uncorrelated with each other. This is related to the classic Markov assumptions and ladder approximations made to describe optical and laser beam propagation through the turbulent atmosphere.[15]

Here the lower limit of the integral occurs in the absence of inhomogeneities. It is defined in terms of the depth-integrated second moment of the incident field,

$$W_i(\boldsymbol{\rho}|\mathbf{r_0}) = \int_0^\infty \frac{1}{d(z)}|\Psi_i(\mathbf{r}|\mathbf{r_0})|^2\,\mathrm{d}z = \sum_n W_i^{(n)}(\boldsymbol{\rho}|\mathbf{r_0}), \quad (31)$$

where

$$W_i^{(n)}(\boldsymbol{\rho}|\mathbf{r_0}) = \frac{2\pi}{d^2(z_0)}|u_n(z_0)|^2\frac{1}{\rho|\xi_n|}\mathrm{e}^{-2\Im\{\xi_n\}\rho}, \quad (32)$$

decays as a function of horizontal range as a result of cylindrical spreading in the waveguide through the $1/\rho$ dependence and absorption loss in the water column and bottom through $\mathrm{e}^{-2\Im\{\xi_n\}\rho}$.

Integrating Eq. (30), we then have

$$\langle W_T^{(n)}(\boldsymbol{\rho}|\mathbf{r_0})\rangle = W_i^{(n)}(\boldsymbol{\rho}|\mathbf{r_0})\mathrm{e}^{\int_0^\rho(\mu_n(\rho_s)-2\Im\{\nu_n(\rho_s)\})\mathrm{d}\rho_s}. \quad (33)$$

Summing the contributions from all modes, the depth-integrated second moment of the forward field in the randomly inhomogeneous waveguide is

$$\langle W_T(\boldsymbol{\rho}|\mathbf{r_0})\rangle = \sum_n \langle W_T^{(n)}(\boldsymbol{\rho}|\mathbf{r_0})\rangle = \sum_n W_i^{(n)}(\boldsymbol{\rho}|\mathbf{r_0})\mathrm{e}^{\int_0^\rho(\mu_n(\rho_s)-2\Im\{\nu_n(\rho_s)\})\mathrm{d}\rho_s}, \quad (34)$$

which decays with range due to (1) spreading and absorption loss in the incident field and (2) modal extinction from scattering by inhomogeneities as seen in the argument of the exponential. It is important to point out that Eq. (34) accounts for possible range-dependent variation in the expected scattering properties of the inhomogeneities.

C. Second moment and variance of the forward field

Here we develop analytic expressions for the second moment and variance of the forward field after propagating through a random inhomogeneous waveguide to a single receiver.

The forward field $\Psi_T(\mathbf{r}|\mathbf{r_0})$ received at $\mathbf{r}$ after propagation through a waveguide containing inhomogeneities can be expressed as a modal sum of the form

$$\Psi_T(\mathbf{r}|\mathbf{r_0}) = \sum_n \chi_T^{(n)}(\boldsymbol{\rho}|\mathbf{r_0})u_n(z), \quad (35)$$

where the range-dependent part of the field given by $\chi_T^{(n)}(\boldsymbol{\rho}|\mathbf{r_0})$ is separated from the depth-dependent part given by the mode amplitude $u_n(z)$ at receiver depth z.

The mean forward field from Eq. (35) is

$$\langle\Psi_T(\mathbf{r}|\mathbf{r_0})\rangle = \sum_n \langle\chi_T^{(n)}(\boldsymbol{\rho}|\mathbf{r_0})\rangle u_n(z), \quad (36)$$

where the expectation is taken over the range-dependent part of the field. The mean forward field is then expressed in terms of the modes of the incident field in the medium without random inhomogeneities. This follows from successive application of Green's theorem to describe multiple forward scattering through consecutive single scatter shells, as has been noted in previous sections and will be discussed again in Sec. IV. From Eqs. (18), (22), and (36), we observe that $\chi_T^{(n)}(\boldsymbol{\rho}|\mathbf{r_0})$ has a mean given by

$$\langle\chi_T^{(n)}(\boldsymbol{\rho}|\mathbf{r_0})\rangle = 4\pi\frac{i}{d(z_0)\sqrt{8\pi}}e^{-i\pi/4}u_n(z_0)\frac{\mathrm{e}^{i\xi_n\rho}}{\sqrt{\xi_n\rho}}\mathrm{e}^{i\int_0^\rho\nu_n(\rho_s)\mathrm{d}\rho_s}. \quad (37)$$

The second moment of the forward field at receiver $\mathbf{r}$ is

$$\langle|\Psi_T(\mathbf{r}|\mathbf{r_0})|^2\rangle = \langle\Psi_T(\mathbf{r}|\mathbf{r_0})\Psi_T^*(\mathbf{r}|\mathbf{r_0})\rangle$$
$$= \sum_n \sum_m \langle\chi_T^{(n)}(\boldsymbol{\rho}|\mathbf{r_0})\chi_T^{*(m)}(\boldsymbol{\rho}|\mathbf{r_0})\rangle u_n(z)u_m^*(z). \quad (38)$$

We assume that the modes are statistically independent. This is a valid assumption beyond several waveguide depths in range after significant multiple forward scattering. The cross-modal coherence of the range-dependent part of the forward field can then be expressed as

$$\langle\chi_T^{(n)}(\boldsymbol{\rho}|\mathbf{r_0})\chi_T^{*(m)}(\boldsymbol{\rho}|\mathbf{r_0})\rangle$$
$$= \langle\chi_T^{(n)}(\boldsymbol{\rho}|\mathbf{r_0})\rangle\langle\chi_T^{*(m)}(\boldsymbol{\rho}|\mathbf{r_0})\rangle + \delta_{nm}\,\mathrm{Var}(\chi_T^{(n)}(\boldsymbol{\rho}|\mathbf{r_0})), \quad (39)$$

where

$$\mathrm{Var}(\chi_T^{(n)}(\boldsymbol{\rho}|\mathbf{r}_0)) = \langle|\chi_T^{(n)}(\boldsymbol{\rho}|\mathbf{r_0})|^2\rangle - |\langle\chi_T^{(n)}(\boldsymbol{\rho}|\mathbf{r_0})\rangle|^2, \quad (40)$$

is the variance of the range-dependent part of the forward field for each mode n. Substituting Eq. (39) into Eq. (38), the expected intensity of the forward field becomes

$$\langle|\Psi_T(\mathbf{r}|\mathbf{r_0})|^2\rangle$$
$$= \sum_n \sum_m \langle\chi_T^{(n)}(\boldsymbol{\rho}|\mathbf{r_0})\rangle\langle\chi_T^{*(m)}(\boldsymbol{\rho}|\mathbf{r_0})\rangle u_n(z)u_m^*(z),$$
$$+ \sum_n \mathrm{Var}(\chi_T^{(n)}(\boldsymbol{\rho}|\mathbf{r_0}))|u_n(z)|^2, \quad (41)$$

$$= |\langle\Psi_T(\mathbf{r}|\mathbf{r_0})\rangle|^2 + \mathrm{Var}(\Psi_T(\mathbf{r}|\mathbf{r_0})). \quad (42)$$

The last equality follows from the use of Eq. (36) and the definition of the variance of the forward field, where

$$\mathrm{Var}(\Psi_T(\mathbf{r}|\mathbf{r_0})) = \langle|\Psi_T(\mathbf{r}|\mathbf{r_0})|^2\rangle - |\langle\Psi_T(\mathbf{r}|\mathbf{r_0})\rangle|^2 \quad (43)$$

$$= \sum_n \mathrm{Var}(\chi_T^{(n)}(\boldsymbol{\rho}|\mathbf{r_0}))|u_n(z)|^2. \quad (44)$$

In Eq. (42), the second moment of the forward field is expressed as a sum of the square of the mean field $|\langle\Psi_T(\mathbf{r}|\mathbf{r_0})\rangle|^2$ which is proportional to the coherent direct wave intensity, and the variance of the forward field $\mathrm{Var}(\Psi_T(\mathbf{r}|\mathbf{r_0}))$, which is proportional to the incoherent direct wave intensity.

The second moment and variance of the forward field at a single receiver can be calculated using Eqs. (40), (41), and (44) from knowledge of the mean and second moment of $\chi_T^{(n)}(\boldsymbol{\rho}|\mathbf{r_0})$ for each mode n, where the mean is given in Eq. (37), and the second moment can be computed from the depth-integrated second moment of the forward field.

The depth-integrated second moment of the forward field can be written as

$$\langle W_T(\boldsymbol{\rho}|\mathbf{r_0})\rangle = \int_0^\infty \frac{1}{d(z)}\langle|\Psi_T(\mathbf{r}|\mathbf{r_0})|^2\rangle \mathrm{d}z \quad (45)$$

$$= \sum_n \langle|\chi_T^{(n)}(\boldsymbol{\rho}|\mathbf{r_0})|^2\rangle, \quad (46)$$

where modal orthogonality,

$$\int_0^\infty \frac{1}{d(z)}u_m(z)u_n(z)\mathrm{d}z = \delta_{nm} \quad (47)$$

collapses the double modal sum in Eq. (38) into the final single sum.

The second moment of $\chi_T^{(n)}(\boldsymbol{\rho}|\mathbf{r_0})$ for each mode n can be obtained from the depth-integrated second moment of the forward field using Eq. (46). In Sec. III B, we derive analytic expressions for $W_T(\boldsymbol{\rho}|\mathbf{r_0})$ and hence $\langle|\chi_T^{(n)}(\boldsymbol{\rho}|\mathbf{r_0})|^2\rangle$.

Comparing Eq. (34) with Eq. (46), the second moment of $\chi_T^{(n)}(\boldsymbol{\rho}|\mathbf{r_0})$ becomes

$$\langle|\chi_T^{(n)}(\boldsymbol{\rho}|\mathbf{r_0})|^2\rangle = W_i^{(n)}(\boldsymbol{\rho}|\mathbf{r_0})\mathrm{e}^{\int_0^\rho(\mu_n(\rho_s)-2\Im\{\nu_n(\rho_s)\})\mathrm{d}\rho_s}. \quad (48)$$

The variance of the forward field at any receiver depth in the waveguide can now be analytically expressed as the single modal sum,

$$\mathrm{Var}(\Psi_T(\mathbf{r}|\mathbf{r_0}))$$
$$= \sum_n W_i^{(n)}(\boldsymbol{\rho}|\mathbf{r_0})|u_n(z)|^2\mathrm{e}^{-\int_0^\rho 2\Im\{\nu_n(\rho_s)\}\mathrm{d}\rho_s}\left(\mathrm{e}^{\int_0^\rho\mu_n(\rho_s)\mathrm{d}\rho_s} - 1\right),$$
$$(49)$$

from Eqs. (37), (40), (44), and (48).

IV. DERIVATION OF DIRECT WAVE MOMENTS FOR A SHELL OF INHOMOGENEITIES IN TERMS OF EXPONENTIAL COEFFICIENTS OF MODAL ATTENUATION, DISPERSION, AND FIELD VARIANCE

Here we derive the first and second statistical moments of the direct wave after propagation through an elemental shell of inhomogeneities within which the single scatter approximation is valid. These moments are expressed in terms of exponential coefficients of modal attenuation, dispersion, and field variance. It is shown that both the first and depth-integrated second moment can be written as a single sum over the waveguide modes as long as the shell width within which the single scatter approximation is valid is sufficiently large for modal decoupling to occur.

A. Modal attenuation and dispersion coefficients

Here it is shown that the mean scattered field contribution to the direct wave from a cylindrical shell of inhomogeneities is given by the single modal sum of Eq. (13), which is a linear function of shell thickness and the horizontal wave number change ν_n of mode n, under widely satisfied conditions.

The *mean* scattered field from any shell that contributes to the direct wave is found by taking the expected value of Eq. (9),

$$\langle \Phi_s(\mathbf{r}|\mathbf{r_0},\Delta\rho_s(\rho_s))\rangle$$
$$= \sum_m \sum_n \int_0^\infty \mathrm{d}z_t \int_{\rho_s-\Delta\rho_s/2}^{\rho_s+\Delta\rho_s/2} \mathrm{d}\rho_t \int_{\phi-\phi_F/2}^{\phi+\phi_F/2} \rho_t\,\mathrm{d}\phi_t\, \mathrm{e}^{i\xi_m[\rho\rho_t/2(\rho-\rho_t)](\phi_t-\phi)^2}$$
$$\times \frac{2\pi}{k}\frac{i}{d(z_0)d(z_t)}\frac{1}{\sqrt{\xi_m\xi_n(\rho-\rho_t)\rho_t}}u_m(z)u_n(z_0)\mathrm{e}^{i\pi/4}\mathrm{e}^{i\xi_m\rho}\mathrm{e}^{i(\xi_n-\xi_m)\rho_t}$$
$$\times[N_m^{(1)}N_n^{(1)}\mathrm{e}^{i(\gamma_m+\gamma_n)z_t}\langle s_{\mathbf{r_t}}(\pi-\alpha_m,\beta_s(\phi,\phi_t);\alpha_n,\phi_t)\rangle - N_m^{(2)}N_n^{(1)}\mathrm{e}^{i(-\gamma_m+\gamma_n)z_t}\langle s_{\mathbf{r_t}}(\alpha_m,\beta_s(\phi,\phi_t);\alpha_n,\phi_t)\rangle$$
$$- N_m^{(1)}N_n^{(2)}\mathrm{e}^{i(\gamma_m-\gamma_n)z_t}\langle s_{\mathbf{r_t}}(\pi-\alpha_m,\beta_s(\phi,\phi_t);\pi-\alpha_n,\phi_t)\rangle + N_m^{(2)}N_n^{(2)}\mathrm{e}^{i(-\gamma_m-\gamma_n)z_t}\langle s_{\mathbf{r_t}}(\alpha_m,\beta_s(\phi,\phi_t);\pi-\alpha_n,\phi_t)\rangle]. \quad (50)$$

So long as the single scatter approximation is valid within the given inhomogeneous shell, the Green functions from the source to the shell and from the shell to the receiver can be treated deterministically. The expectation values then operate only on the scatter functions of random shell inhomogeneities, as noted in the derivation of Eq. (8) of Ref. 5.

We assume that the inhomogeneities only need to obey a horizontally stationary random process within the Fresnel width in any given shell, but not over larger separations. The expected scattering properties of the medium then may be both range and azimuth dependent in the present formulation. Also, the scattering properties need not be stationary over depth. This makes it possible to adopt the abbreviated notation $\langle s_{\mathbf{r_t}}(\alpha,\beta;\alpha_i,\beta_i)\rangle = \langle s_{\rho_s,z_t}(\alpha,\beta;\alpha_i,\beta_i)\rangle$ within the Fresnel width in any given shell.

Given the assumed statistical stationarity of the scatter function density within the integrand of Eq. (50), the integral over azimuth can be evaluated by conventional stationary phase methods to yield

$$\langle \Phi_s(\mathbf{r}|\mathbf{r_0},\Delta\rho_s(\rho_s))\rangle = \sum_m \sum_n \int_0^\infty \mathrm{d}z_t \int_{\rho_s-\Delta\rho_s/2}^{\rho_s+\Delta\rho_s/2} \mathrm{d}\rho_t \sqrt{\frac{2\pi(\rho-\rho_t)}{\xi_m\rho\rho_t}}\rho_t\frac{2\pi}{k}\frac{i}{d(z_0)d(z_t)}\frac{1}{\sqrt{\xi_m\xi_n(\rho-\rho_t)\rho_t}}u_m(z)u_n(z_0)\mathrm{e}^{i\pi/4}\mathrm{e}^{i\xi_m\rho}\mathrm{e}^{i(\xi_n-\xi_m)\rho_t}$$
$$\times[N_m^{(1)}N_n^{(1)}\mathrm{e}^{i(\gamma_m+\gamma_n)z_t}\langle s_{\rho_s,z_t}(\pi-\alpha_m,\phi;\alpha_n,\phi)\rangle - N_m^{(2)}N_n^{(1)}\mathrm{e}^{i(-\gamma_m+\gamma_n)z_t}\langle s_{\rho_s,z_t}(\alpha_m,\phi;\alpha_n,\phi)\rangle$$
$$- N_m^{(1)}N_n^{(2)}\mathrm{e}^{i(\gamma_m-\gamma_n)z_t}\langle s_{\rho_s,z_t}(\pi-\alpha_m,\phi;\pi-\alpha_n,\phi)\rangle + N_m^{(2)}N_n^{(2)}\mathrm{e}^{i(-\gamma_m-\gamma_n)z_t}\langle s_{\rho_s,z_t}(\alpha_m,\phi;\pi-\alpha_n,\phi)\rangle]. \quad (51)$$

Equation (51) is then integrated in range ρ_t over the shell width $\Delta\rho_s$ about ρ_s. Applying exponential integration of the form

$$\int_{\rho_s-\Delta\rho_s/2}^{\rho_s+\Delta\rho_s/2} \mathrm{e}^{i(\xi_n-\xi_m)\rho_t}\mathrm{d}\rho_t = \mathrm{e}^{i(\xi_n-\xi_m)\rho_s}\Delta\rho_s\,\mathrm{sinc}\left((\xi_n-\xi_m)\frac{\Delta\rho_s}{2}\right), \quad (52)$$

leads to

$$\langle \Phi_s(\mathbf{r}|\mathbf{r_0},\Delta\rho_s(\rho_s))\rangle = \sum_m \sum_n \int_0^\infty \mathrm{d}z_t \frac{2\pi}{k}\frac{i}{d(z_0)d(z_t)}\frac{\sqrt{2\pi}}{\xi_m}\frac{\mathrm{e}^{i\xi_m\rho}}{\sqrt{\xi_n\rho}}u_m(z)u_n(z_0)\mathrm{e}^{i\pi/4}\mathrm{e}^{i(\xi_n-\xi_m)\rho_s}\,\mathrm{sinc}\left((\xi_n-\xi_m)\frac{\Delta\rho_s}{2}\right)$$
$$\times[N_m^{(1)}N_n^{(1)}\mathrm{e}^{i(\gamma_m+\gamma_n)z_t}\langle s_{\rho_s,z_t}(\alpha_m,\phi;\pi-\alpha_n,\phi)\rangle - N_m^{(2)}N_n^{(1)}\mathrm{e}^{i(-\gamma_m+\gamma_n)z_t}\langle s_{\rho_s,z_t}(\alpha_m,\phi;\alpha_n,\phi)\rangle$$
$$- N_m^{(1)}N_n^{(2)}\mathrm{e}^{i(\gamma_m-\gamma_n)z_t}\langle s_{\rho_s,z_t}(\pi-\alpha_m,\phi;\pi-\alpha_n,\phi)\rangle + N_m^{(2)}N_n^{(2)}\mathrm{e}^{i(-\gamma_m-\gamma_n)z_t}\langle s_{\rho_s,z_t}(\pi-\alpha_m,\phi;\alpha_n,\phi)\rangle]\Delta\rho_s. \quad (53)$$

The field scattered from the shell given in Eq. (53) reduces to a single modal sum by modal orthogonality for compact scatterers that obey a stationary random process in depth throughout the waveguide, as will be shown in Sec. IV A 1. For general inhomogeneities with arbitrary depth distribution, a single modal sum is obtained under the widely satisfied conditions derived in Sec. IV A 2.

1. Compact inhomogeneities that obey a stationary random process in depth

Acoustically compact inhomogeneities, those small compared to the wavelength, or inhomogeneities following the sonar equation in a waveguide[54] that obey a stationary random process in depth have expected scatter function densities that are independent of both direction and depth, so that $\langle s_{\rho_s,z_t}(\alpha,\beta;\alpha_i,\beta_i)\rangle = \langle s_0(\rho_s)\rangle$, can be factored from Eq. (53).

Equation (53) for the mean scattered field from the shell then reduces to the single modal sum,

$$\langle \Phi_s(\mathbf{r}|\mathbf{r_0},\Delta\rho_s(\rho_s))\rangle = \sum_n 4\pi \frac{i}{d(z_0)\sqrt{8\pi}} e^{-i\pi/4} u_n(z) u_n(z_0) \times \frac{e^{i\xi_n\rho}}{\sqrt{\xi_n\rho}} i\left(\frac{2\pi}{k\xi_n}\langle s_0(\rho_s)\rangle\right)\Delta\rho_s, \tag{54}$$

by modal orthogonality, via Eq. (47).

The mean scattered field from the shell can then be expressed in terms of the incident field,

$$\langle \Phi_s(\mathbf{r}|\mathbf{r_0},\Delta\rho_s(\rho_s))\rangle = \sum_n \Phi_i^{(n)}(\mathbf{r}|\mathbf{r_0}) i\nu_n(\rho_s)\Delta\rho_s, \tag{55}$$

by substituting Eq. (18) into Eq. (54), where,

$$\nu_n(\rho_s) = \frac{2\pi}{k\xi_n}\langle s_0(\rho_s)\rangle. \tag{56}$$

Equation (56) defines the horizontal wave number change of each mode due to scattering from 3-D compact inhomogeneities that obey a stationary random process over waveguide depth. For mode 1, $\xi_1 \approx k$ is an excellent approximation that makes the corresponding wave number change approximately equal to that found for propagation through inhomogeneities in free space.[11] Higher-order modes travel at steeper elevation angles, have longer propagation paths, interact more with the inhomogeneities, and so experience greater attenuation and dispersion. This is seen in Eq. (56), where modal attenuation and dispersion coefficients are inversely proportional to the horizontal wave number of the given mode.

2. General inhomogeneities with arbitrary depth dependence

All terms in Eq. (53) are proportional to $\mathrm{sinc}[(\xi_n-\xi_m)(\Delta\rho_s/2)]$, which is unity for diagonal terms, where $n=m$, but becomes negligibly small for off-diagonal terms, where $n\neq m$, when the shell thickness is large enough that $\mathrm{sinc}[(\xi_n-\xi_m)(\Delta\rho_s/2)]_{\Delta\rho_s=\Delta\rho_{\max}} \ll 1$ or equivalently the condition

$$\Delta\rho_{\max}(\sin\alpha_n - \sin\alpha_m) \gg \lambda, \tag{57}$$

is satisfied.[51]

The mean forward field then reduces to a single modal sum when the maximum shell thickness $\Delta\rho_{\max}$ for which the single scatter approximation is valid satisfies condition (57). A similar condition has been presented by Frankenthal and Beran[40] for forward propagation and in Ref. 5 for reverberation.

Condition (57) is most easily interpreted in terms of diffraction theory by considering plane wave propagation through an aperture of length $\Delta\rho_{\max}\cos(\pi/2-\alpha_n)$, which equals the projected length of $\Delta\rho_{\max}$ in the equivalent plane wave propagation direction of mode n. In the forward azimuth, condition (57) requires the difference in projected apertures between modes n and m for $n\neq m$ to be much larger than the wavelength. For typical low-frequency applications, $\Delta\rho_{\max}$ will range in scale from the minumum channel depth, or water depth in Continental Shelf environments, to at most one order of magnitude larger.

The mean scattered field from the shell, under condition (57), is then the single modal sum

$$\langle \Phi_s(\mathbf{r}|\mathbf{r_0},\Delta\rho_s(\rho_s))\rangle = \sum_n 4\pi \frac{i}{d(z_0)\sqrt{8\pi}} e^{-i\pi/4} u_n(z) u_n(z_0) \frac{e^{i\xi_n\rho}}{\sqrt{\xi_n\rho}} \times i\int \frac{1}{d(z_t)}\frac{2\pi}{k}\frac{1}{\xi_n}[(N_n^{(1)})^2 e^{i2\gamma_n z_t}\langle s_{\rho_s,z_t}(\pi-\alpha_n,\phi;\alpha_n,\phi)\rangle - N_n^{(2)}N_n^{(1)}\langle s_{\rho_s,z_t}(\alpha_n,\phi;\alpha_n,\phi)\rangle - N_n^{(1)}N_n^{(2)}\langle s_{\rho_s,z_t}(\pi-\alpha_n,\phi;\pi-\alpha_n,\phi)\rangle + (N_n^{(2)})^2 e^{-i2\gamma_n z_t}\langle s_{\rho_s,z_t}(\alpha_n,\phi;\pi-\alpha_n,\phi)\rangle]\mathrm{d}z_t\Delta\rho_{\max}. \tag{58}$$

Substituting Eq. (18) into Eq. (58), the mean scattered field from the shell is then expressible in terms of the incident field via

$$\langle \Phi_s(\mathbf{r}|\mathbf{r_0},\Delta\rho_s(\rho_s))\rangle = \sum_n \Phi_i^{(n)}(\mathbf{r}|\mathbf{r_0}) i\nu_n(\rho_s)\Delta\rho_{\max}, \tag{59}$$

where

$$\nu_n(\rho_s) = \int_0^\infty \frac{2\pi}{k}\frac{1}{\xi_n}\frac{1}{d(z_t)}[(N_n^{(1)})^2 \mathrm{e}^{i2\gamma_n z_t}\langle s_{\rho_s,z_t}(\pi-\alpha_n,\phi;\alpha_n,\phi)\rangle - N_n^{(2)}N_n^{(1)}\langle s_{\rho_s,z_t}(\alpha_n,\phi;\alpha_n,\phi)\rangle - N_n^{(1)}N_n^{(2)}\langle s_{\rho_s,z_t}(\pi-\alpha_n,\phi;\pi-\alpha_n,\phi)\rangle + (N_n^{(2)})^2\mathrm{e}^{-i2\gamma_n z_t}\langle s_{\rho_s,z_t}(\alpha_n,\phi;\pi-\alpha_n,\phi)\rangle]\mathrm{d}z_t. \quad (60a)$$

Equation (60) defines the horizontal wave number change of the nth mode due to scattering in a waveguide with general inhomogeneities arbitrarily distributed in depth. For compact scatterers that follow a stationary random process in depth, Eq. (60) reduces to Eq. (56).

By applying the generalized waveguide extinction theorem, Eq. (20) of Ref. 6 with $x=0$, the attenuation coefficient due to scattering for each mode n can be expressed in terms of the waveguide extinction cross-section σ_n of the medium inhomogeneities as,

$$\Im\{\nu_n(\rho_s)\} = \frac{1}{V_c}\int_0^\infty \frac{1}{2}\frac{1}{d(z_t)}\frac{1}{|u_n(z_t)|^2}\langle\sigma_n\rangle \mathrm{d}z_t. \quad (60b)$$

Equation (60b) is convenient because it can be used to compute attenuation due to scattering in the randomly inhomogeneous waveguide directly from a knowledge of the expected waveguide extinction cross-section density of the medium's inhomogeneities.

B. Coefficient of modal field variance

We now derive an analytic expression for the exponential coefficient of modal field variance μ_n for a random inhomogeneous waveguide. Here, we show that Eq. (25) is valid for scattering from a cylindrical shell element containing inhomogeneities under widely applicable conditions.

Again employing the single scatter approximation within a shell of inhomogeneities of thickness $\Delta\rho_s$ at range ρ_s, from Eq. (9), the second moment of the direct-wave component of the scattered field from the shell is found to be

$$\begin{aligned}
&\langle|\Phi_s(\mathbf{r}|\mathbf{r_0},\Delta\rho_s(\rho_s))|^2\rangle \\
&= \Bigg\langle \sum_m\sum_n\sum_{m'}\sum_{n'}\int_0^\infty \mathrm{d}z_t\int_0^\infty \mathrm{d}z_{t'}\int_{\rho_s-\Delta\rho_s/2}^{\rho_s+\Delta\rho_s/2}\mathrm{d}\rho_t\int_{\rho_s-\Delta\rho_s/2}^{\rho_s+\Delta\rho_s/2}\mathrm{d}\rho_{t'} \\
&\times \int_{\phi-\phi_F/2}^{\phi+\phi_F/2}\rho_t\,\mathrm{d}\phi_t\,\mathrm{e}^{i\xi_m[\rho\rho_t/2(\rho-\rho_t)](\phi_t-\phi)^2}\int_{\phi-\phi_F/2}^{\phi+\phi_F/2}\rho_{t'}\,\mathrm{d}\phi_{t'}\,\mathrm{e}^{-i\xi_{m'}[\rho\rho_{t'}/2(\rho-\rho_{t'})](\phi_{t'}-\phi)^2} \\
&\times \frac{4\pi^2}{k(z_t)k(z_{t'})d(z_t)d(z_{t'})}\frac{1}{d^2(z_0)}\frac{1}{\sqrt{\xi_m\xi_{m'}^*}}\frac{1}{\sqrt{\xi_n\xi_{n'}^*}}\frac{1}{\sqrt{(\rho-\rho_t)(\rho-\rho_{t'})\rho_t\rho_{t'}}}u_m(z)u_{m'}^*(z)u_n(z_0)u_{n'}^*(z_0) \\
&\times [N_m^{(1)}(z_t)N_n^{(1)}(z_t)\mathrm{e}^{i\Re(\gamma_m+\gamma_n)z_t}s_{\mathbf{r_t}}(\pi-\alpha_m,\beta_s(\phi,\phi_t);\alpha_n,\phi_t) \\
&- N_m^{(2)}(z_t)N_n^{(1)}(z_t)\mathrm{e}^{i\Re(-\gamma_m+\gamma_n)z_t}s_{\mathbf{r_t}}(\alpha_m,\beta_s(\phi,\phi_t);\alpha_n,\phi_t) \\
&- N_m^{(1)}(z_t)N_n^{(2)}(z_t)\mathrm{e}^{i\Re(\gamma_m-\gamma_n)z_t}s_{\mathbf{r_t}}(\pi-\alpha_m,\beta_s(\phi,\phi_t);\pi-\alpha_n,\phi_t) \\
&+ N_m^{(2)}(z_t)N_n^{(2)}(z_t)\mathrm{e}^{i\Re(-\gamma_m-\gamma_n)z_t}s_{\mathbf{r_t}}(\alpha_m,\beta_s(\phi,\phi_t);\pi-\alpha_n,\phi_t)] \\
&\times[N_{m'}^{*(1)}(z_{t'})N_{n'}^{*(1)}(z_{t'})\mathrm{e}^{i\Re(-\gamma_{m'}-\gamma_{n'})z_{t'}}s_{\mathbf{r_{t'}}}(\pi-\alpha_{m'},\beta_s(\phi,\phi_{t'});\alpha_{n'},\phi_{t'}) \\
&- N_{m'}^{*(2)}(z_{t'})N_{n'}^{*(1)}(z_{t'})\mathrm{e}^{i\Re(\gamma_{m'}-\gamma_{n'})z_{t'}}s_{\mathbf{r_{t'}}}(\alpha_{m'},\beta_s(\phi,\phi_{t'});\alpha_{n'},\phi_{t'}) \\
&- N_{m'}^{*(1)}(z_{t'})N_{n'}^{*(2)}(z_{t'})\mathrm{e}^{i\Re(-\gamma_{m'}+\gamma_{n'})z_{t'}}s_{\mathbf{r_{t'}}}(\pi-\alpha_{m'},\beta_s(\phi,\phi_{t'});\pi-\alpha_{n'},\phi_{t'}) \\
&+ N_{m'}^{*(2)}(z_{t'})N_{n'}^{*(2)}(z_{t'})\mathrm{e}^{i\Re(\gamma_{m'}+\gamma_{n'})z_{t'}}s_{\mathbf{r_{t'}}}(\alpha_{m'},\beta_s(\phi,\phi_{t'});\pi-\alpha_{n'},\phi_{t'})] \\
&\times \mathrm{e}^{i\Re\{\xi_m(\rho-\rho_t)-\xi_{m'}(\rho-\rho_{t'})\}}\mathrm{e}^{i\Re\{\xi_n\rho_t-\xi_{n'}\rho_{t'}\}} \\
&\times \mathrm{e}^{-\Im\{\xi_m+\xi_{m'}\}(\rho-\rho_t)}\mathrm{e}^{-\Im\{\xi_n+\xi_{n'}\}\rho_t}\mathrm{e}^{-\Im\{(\gamma_m+\gamma_n)z_t+(\gamma_{m'}+\gamma_{n'})z_{t'}\}}\Bigg\rangle, \quad (61)
\end{aligned}$$

where the expectation values only operate on scatter function density products following the assumed statistical decorrelation between inhomogenieties in adjacent shells, as discussed previously.

Equation (61) can only be further evaluated if the cross-correlation of the scatter function density of the medium at $\mathbf{r_t}$ and $\mathbf{r_{t'}}$ is known. We will assume that the inhomogeneities obey a stationary random process within the Fresnel width in each single scatter shell. The second moment of the scattered field contribution to the direct wave from a given shell then depends on whether the cross-range coherence length l_y of the random process, defined in Appendix A, is greater or less than the Fresnel width Y_{F} at the given shell range ρ_s. Scatterers are fully correlated across the active region or Fresnel width of the shell when $l_y > Y_{\mathrm{F}}$, but are uncorrelated in range and possibly depth. When, $l_y < Y_{\mathrm{F}}$, fluctuations arise from scatterers uncorrelated in both range, cross-range, and possibly depth.

1. Fully correlated scatterers within the Fresnel width

Inhomogeneities within the shell at range ρ_s are now assumed to satisfy $l_y > Y_{\mathrm{F}}(\rho,\rho_s)$ or $|\rho_s-\rho/2| > (\rho/2)\sqrt{1-4l_y^2/\lambda\rho}$. Their scatter function densities are then taken to be fully correlated in cross-range within the active region of the shell, but become uncorrelated when their separation in range exceeds the coherence length $l_x(\rho_s,z_t,z_{t'})$, defined in Appendix A, so that

$$\begin{aligned}
&\langle s_{\mathbf{r_t}}(\pi-\alpha_m,\beta_s(\phi,\phi_t);\alpha_n,\phi_t)s^*_{\mathbf{r_{t'}}}(\pi-\alpha_{m'},\beta_s(\phi,\phi_{t'});\alpha_{n'},\phi_{t'})\rangle \\
&\quad\approx l_x(\rho_s,z_t,z_t')[\langle s_{\rho_s,z_t}(\pi-\alpha_m,\beta_s(\phi,\phi_t);\alpha_n,\phi_t)s^*_{\rho_s,z_{t'}}(\pi-\alpha_{m'},\beta_s(\phi,\phi_{t'});\alpha_{n'},\phi_{t'})\rangle \\
&\quad-\langle s_{\rho_s,z_t}(\pi-\alpha_m,\beta_s(\phi,\phi_t);\alpha_n,\phi_t)\rangle\langle s^*_{\rho_s,z_{t'}}(\pi-\alpha_{m'},\beta_s(\phi,\phi_{t'});\alpha_{n'},\phi_{t'})\rangle]\delta(x_t-x_{t'}) \\
&\quad+\langle s_{\rho_s,z_t}(\pi-\alpha_m,\beta_s(\phi,\phi_t);\alpha_n,\phi_t)\rangle\langle s^*_{\rho_s,z_{t'}}(\pi-\alpha_{m'},\beta_s(\phi,\phi_{t'});\alpha_{n'},\phi_{t'})\rangle. \qquad (62)
\end{aligned}$$

The second moment of the scattered field, obtained by substituting Eq. (62) into Eq. (61), has 16 similar terms, the first of which is

$$\begin{aligned}
&\langle|\Phi_s(\mathbf{r}|\mathbf{r}_0,\Delta\rho_s(\rho_s))|^2\rangle_1 \\
&\quad=\sum_m\sum_n\sum_{m'}\sum_{n'}\int_0^\infty \mathrm{d}z_t\int_0^\infty \mathrm{d}z_{t'}\int_{\rho_s-\Delta\rho_s/2}^{\rho_s+\Delta\rho_s/2}\mathrm{d}\rho_t l_x(\rho_s,z_t,z_{t'}) \\
&\quad\times\int_{\phi-\phi_F/2}^{\phi+\phi_F/2}\rho_t\,\mathrm{d}\phi_t\,\mathrm{e}^{i\xi_m[\rho\rho_t/2(\rho-\rho_t)](\phi_t-\phi)^2}\int_{\phi-\phi_F/2}^{\phi+\phi_F/2}\rho_t\,\mathrm{d}\phi_{t'}\,\mathrm{e}^{-i\xi_{m'}[\rho\rho_t/2(\rho-\rho_t)](\phi_{t'}-\phi)^2} \\
&\quad\times\frac{4\pi^2}{k(z_t)k(z_{t'})d(z_t)d(z_{t'})}\frac{1}{d^2(z_0)}\frac{1}{\sqrt{\xi_m\xi^*_{m'}}}\frac{1}{\sqrt{\xi_n\xi^*_{n'}}}\frac{1}{(\rho-\rho_t)\rho_t} \\
&\quad\times u_m(z)u^*_{m'}(z)u_n(z_0)u^*_{n'}(z_0)N_m^{(1)}(z_t)N_n^{(1)}(z_t)N_{m'}^{*(1)}(z_{t'})N_{n'}^{*(1)}(z_{t'}) \\
&\quad\times[\langle s_{\rho_s,z_t}(\pi-\alpha_m,\beta_s(\phi,\phi_t);\alpha_n,\phi_t)s^*_{\rho_s,z_{t'}}(\pi-\alpha_{m'},\beta_s(\phi,\phi_{t'});\alpha_{n'},\phi_{t'})\rangle \\
&\quad-\langle s_{\rho_s,z_t}(\pi-\alpha_m,\beta_s(\phi,\phi_t);\alpha_n,\phi_t)\rangle\langle s^*_{\rho_s,z_{t'}}(\pi-\alpha_{m'},\beta_s(\phi,\phi_{t'});\alpha_{n'},\phi_{t'})\rangle] \\
&\quad\times\mathrm{e}^{i\Re\{\xi_m-\xi_{m'}\}(\rho-\rho_t)}\mathrm{e}^{i\Re\{\xi_n-\xi_{n'}\}\rho_t}\mathrm{e}^{-\Im\{\xi_m+\xi_{m'}\}(\rho-\rho_t)}\mathrm{e}^{-\Im\{\xi_n+\xi_{n'}\}\rho_t} \\
&\quad\times\mathrm{e}^{i\Re\{(\gamma_m+\gamma_n)z_t-(\gamma_{m'}+\gamma_{n'})z_{t'}\}}\mathrm{e}^{-\Im\{(\gamma_m+\gamma_n)z_t+(\gamma_{m'}+\gamma_{n'})z_{t'}\}} \\
&\quad+|\langle\Phi_s(\mathbf{r}|\mathbf{r}_0,\Delta\rho_s(\rho_s))\rangle_1|^2, \qquad (63)
\end{aligned}$$

where $|\langle\Phi_s(\mathbf{r}|\mathbf{r_0},\Delta\rho_s(\rho_s))\rangle_1|^2$ is the first term of the square of the mean scattered field from the shell given by Eq. (13).

Since one of our fundamental assumptions is that the *mean* scattered field from a shell must be small in comparison to the incident field, as discussed in the Introduction and Sec. III A, the square of the mean scattered field from the shell must be negligible in the second moment. Since the variance of the scattered field, which is comprised of the remaining terms to the right in Eq. (63), is a statistical quantity that need not depend on the mean, it is not necessarily negligible and must be retained in the second moment, where it is assumed to simply be small compared to the squared magnitude of the incident field. This leads to a consistent asymptotic analysis of the field moments, as we will see in Sec. V A 2.

Since the inhomogeneities are fully correlated over the Fresnel width of the shell, the integral over azimuth in Eq. (63) can be evaluated by stationary phase methods to yield

$$\langle|\Phi_s(\mathbf{r}|\mathbf{r_0},\Delta\rho_s(\rho_s))|^2\rangle_1$$

$$= \sum_m \sum_n \sum_{m'} \sum_{n'} \int_0^\infty \mathrm{d}z_t \int_0^\infty \mathrm{d}z_{t'} \int_{\rho_s-\Delta\rho_s/2}^{\rho_s+\Delta\rho_s/2} \mathrm{d}\rho_t \, l_x(\rho_s,z_t,z_{t'})\rho_t^2 \frac{2\pi(\rho-\rho_t)}{\sqrt{\xi_m\xi_{m'}}\rho\rho_t}$$

$$\times \frac{4\pi^2}{k(z_t)k(z_{t'})d(z_t)d(z_{t'})} \frac{1}{d^2(z_0)} \frac{1}{\sqrt{\xi_m\xi_{m'}^*}} \frac{1}{\sqrt{\xi_n\xi_{n'}^*}} \frac{1}{(\rho-\rho_t)\rho_t}$$

$$\times u_m(z)u_{m'}^*(z)u_n(z_0)u_{n'}^*(z_0)N_m^{(1)}(z_t)N_n^{(1)}(z_t)N_{m'}^{*(1)}(z_{t'})N_{n'}^{*(1)}(z_{t'})$$

$$\times[\langle s_{\rho_s,z_t}(\pi-\alpha_m,\phi;\alpha_n,\phi)s_{\rho_s,z_{t'}}^*(\pi-\alpha_{m'},\phi;\alpha_{n'},\phi)\rangle$$

$$-\langle s_{\rho_s,z_t}(\pi-\alpha_m,\phi;\alpha_n,\phi)\rangle\langle s_{\rho_s,z_{t'}}^*(\pi-\alpha_{m'},\phi;\alpha_{n'},\phi)\rangle]$$

$$\times e^{i\Re\{\xi_m-\xi_{m'}\}(\rho-\rho_t)}e^{i\Re\{\xi_n-\xi_{n'}\}\rho_t}e^{-\Im\{\xi_m+\xi_{m'}\}(\rho-\rho_t)}e^{-\Im\{\xi_n+\xi_{n'}\}\rho_t}$$

$$\times e^{i\Re\{(\gamma_m+\gamma_n)z_t-(\gamma_{m'}+\gamma_{n'})z_{t'}\}}e^{-\Im\{(\gamma_m+\gamma_n)z_t+(\gamma_{m'}+\gamma_{n'})z_{t'}\}}. \tag{64}$$

We next integrate over the shell thickness $\Delta\rho_s$. Applying exponential integration of the form

$$\int_{\rho_s-\Delta\rho_s/2}^{\rho_s+\Delta\rho_s/2} e^{i\Re\{(\xi_n-\xi_{n'})-(\xi_m-\xi_{m'})\}\rho_t}\,\mathrm{d}\rho_t = e^{i\Re\{(\xi_n-\xi_{n'})-(\xi_m-\xi_{m'})\}\rho_s}\,\Delta\rho_s\,\mathrm{sinc}\left(\Re\{(\xi_n-\xi_{n'})-(\xi_m-\xi_{m'})\}\frac{\Delta\rho_s}{2}\right), \tag{65}$$

to Eq. (64), we find

$$\langle|\Phi_s(\mathbf{r}|\mathbf{r_0},\Delta\rho_s(\rho_s))|^2\rangle_1$$

$$= \sum_m \sum_n \sum_{m'} \sum_{n'} \int_0^\infty \mathrm{d}z_t \int_0^\infty \mathrm{d}z_{t'}\, 2\pi l_x(\rho_s,z_t,z_{t'})\frac{1}{\sqrt{\xi_m\xi_{m'}}}$$

$$\times\frac{4\pi^2}{k(z_t)k(z_{t'})d(z_t)d(z_{t'})} \frac{1}{d^2(z_0)} \frac{1}{\sqrt{\xi_m\xi_{m'}^*}} \frac{1}{\sqrt{\xi_n\xi_{n'}^*}} \frac{1}{\rho}$$

$$\times u_m(z)u_{m'}^*(z)u_n(z_0)u_{n'}^*(z_0)N_m^{(1)}(z_t)N_n^{(1)}(z_t)N_{m'}^{*(1)}(z_{t'})N_{n'}^{*(1)}(z_{t'})$$

$$\times[\langle s_{\rho_s,z_t}(\pi-\alpha_m,\phi;\alpha_n,\phi)s_{\rho_s,z_{t'}}^*(\pi-\alpha_{m'},\phi;\alpha_{n'},\phi)\rangle$$

$$-\langle s_{\rho_s,z_t}(\pi-\alpha_m,\phi;\alpha_n,\phi)\rangle\langle s_{\rho_s,z_{t'}}^*(\pi-\alpha_{m'},\phi;\alpha_{n'},\phi)\rangle]$$

$$\times e^{i\Re\{\xi_m-\xi_{m'}\}\rho}e^{i\Re\{(\xi_n-\xi_{n'})-(\xi_m-\xi_{m'})\}\rho_s}e^{-\Im\{\xi_m+\xi_{m'}\}\rho}e^{-\Im\{(\xi_n+\xi_{n'})-(\xi_m+\xi_{m'})\}\rho_s}$$

$$\times e^{i\Re\{(\gamma_m+\gamma_n)z_t-(\gamma_{m'}+\gamma_{n'})z_{t'}\}}e^{-\Im\{(\gamma_m+\gamma_n)z_t+(\gamma_{m'}+\gamma_{n'})z_{t'}\}}$$

$$\times\Delta\rho_s\,\mathrm{sinc}\left(\Re\{(\xi_n-\xi_{n'})-(\xi_m-\xi_{m'})\}\frac{\Delta\rho_s}{2}\right). \tag{66}$$

We next integrate Eq. (66) over receiver depth to obtain the depth-integrated second moment of the scattered field from the shell. Applying modal orthogonality, Eq. (47), to Eq. (66), leads to,

$$\int_0^\infty \frac{1}{d(z)}\langle|\Phi_s(\mathbf{r}|\mathbf{r_0},\Delta\rho_s(\rho_s))|^2\rangle_1\mathrm{d}z$$

$$= \sum_m \sum_n \sum_{n'} \int_0^\infty \mathrm{d}z_t \int_0^\infty \mathrm{d}z_{t'}\, 2\pi\ell_x(\rho_s,z_t,z_{t'})\frac{1}{\xi_m}$$

$$\times \frac{4\pi^2}{k(z_t)k(z_{t'})d(z_t)d(z_{t'})} \frac{1}{d^2(z_0)} \frac{1}{|\xi_m|} \frac{1}{\sqrt{\xi_n\xi_{n'}^*}} \frac{1}{\rho}$$

$$\times u_n(z_0)u_{n'}^*(z_0)N_m^{(1)}(z_t)N_n^{(1)}(z_t)N_m^{*(1)}(z_{t'})N_{n'}^{*(1)}(z_{t'})$$

$$\times[\langle s_{\rho_s,z_t}(\pi-\alpha_m,\phi;\alpha_n,\phi)s_{\rho_s,z_{t'}}^*(\pi-\alpha_m,\phi;\alpha_{n'},\phi)\rangle$$

$$-\langle s_{\rho_s,z_t}(\pi-\alpha_m,\phi;\alpha_n,\phi)\rangle\langle s_{\rho_s,z_{t'}}^*(\pi-\alpha_m,\phi;\alpha_{n'},\phi)\rangle]$$

$$\times e^{i\Re\{\xi_n-\xi_{n'}\}\rho_s} e^{-2\Im\{\xi_m\}\rho} e^{-\Im\{(\xi_n+\xi_{n'})-2\xi_m\}\rho_s}$$
$$\times e^{i\Re\{(\gamma_m+\gamma_n)z_t-(\gamma_m+\gamma_{n'})z_{t'}\}} e^{-\Im\{(\gamma_m+\gamma_n)z_t+(\gamma_m+\gamma_{n'})z_{t'}\}}$$
$$\times \Delta\rho_s \mathrm{sinc}\left[\Re\{\xi_n-\xi_{n'}\}\frac{\Delta\rho_s}{2}\right], \tag{67}$$

which has a triple modal sum.

In Eq. (67), terms with $n\neq n'$ have fluctuating phases proportional to $e^{i\Re\{\xi_n-\xi_{n'}\}\rho_s}$. These terms are negligibly small compared to terms for which $n=n'$ when the shell thickness is large enough that

$$\mathrm{sinc}\left[\Re\{\xi_n-\xi_{n'}\}\frac{\Delta\rho_s}{2}\right] \ll 1. \tag{68}$$

This leads to a condition that is identical to Eq. (57) when we let $m=n'$ in Eq. (57).

Under condition (57), the triple sum over the modes in Eq. (67) for the depth-integrated second moment of the scattered field from the shell reduces to a double modal sum,

$$\int_0^\infty \frac{1}{d(z)}\langle|\Phi_s(\mathbf{r}|\mathbf{r_0},\Delta\rho_s(\rho_s))|^2\rangle_1 dz = \sum_m\sum_n \Delta\rho_s \int_0^\infty dz_t \int_0^\infty dz_{t'} 2\pi \frac{\ell_x(\rho_s,z_t,z_{t'})}{\xi_m}\frac{4\pi^2}{k(z_t)k(z_{t'})d(z_t)d(z_{t'})}\frac{1}{d^2(z_0)}\frac{1}{|\xi_m|}\frac{1}{|\xi_n|}\frac{1}{\rho}$$
$$\times |u_n(z_0)|^2 N_m^{(1)}(z_t)N_n^{(1)}(z_t)N_m^{*(1)}(z_{t'})N_n^{*(1)}(z_{t'})\mathrm{Cov}(s_{\rho_s,z_t}(m_u;n_d),s_{\rho_s,z_{t'}}(m_u;n_d))$$
$$\times e^{-2\Im\{\xi_m\}\rho} e^{-2\Im\{\xi_n-\xi_m\}\rho_s} e^{i\Re\{(\gamma_m+\gamma_n)\}(z_t-z_{t'})} e^{-\Im\{(\gamma_m+\gamma_n)\}(z_t+z_{t'})}, \tag{69}$$

where,

$$\mathrm{Cov}(s_{\rho_s,z_t}(m_u;n_d),s_{\rho_s,z_{t'}}(m_u;n_d)) = \langle s_{\rho_s,z_t}(\pi-\alpha_m,\phi;\alpha_n,\phi)s^*_{\rho_s,z_{t'}}(\pi-\alpha_m,\phi;\alpha_n,\phi)\rangle$$
$$-\langle s_{\rho_s,z_t}(\pi-\alpha_m,\phi;\alpha_n,\phi)\rangle\langle s^*_{\rho_s,z_{t'}}(\pi-\alpha_m,\phi;\alpha_n,\phi)\rangle. \tag{70}$$

Combining all sixteen terms leads to the depth-integrated second moment of the scattered field from the shell,

$$\int_0^\infty \frac{1}{d(z)}\langle|\Phi_s(\mathbf{r}|\mathbf{r_0},\Delta\rho_s(\rho_s))|^2\rangle dz = \sum_n \frac{2\pi}{d^2(z_0)}|u_n(z_0)|^2\frac{1}{|\xi_n|\rho}e^{-2\Im\{\xi_n\}\rho}\Delta\rho_s$$
$$\times \sum_m \frac{e^{-2\Im\{\xi_n-\xi_m\}(\rho_s-\rho)}}{|\xi_m|}\int_0^\infty dz_t\int_0^\infty dz_{t'}\frac{\ell_x(\rho_s,z_t,z_{t'})}{\xi_m}\frac{4\pi^2}{k(z_t)k(z_{t'})d(z_t)d(z_{t'})}\mathrm{C}_{s,s}(\rho_s,z_t,z_{t'},m,n), \tag{71}$$

where,

$$\mathrm{C}_{s,s}(\rho_s,z_t,z_{t'},m,n) = [N_m^{(1)}(z_t)N_n^{(1)}(z_t)N_m^{*(1)}(z_{t'})N_n^{*(1)}(z_{t'})e^{i\Re\{(\gamma_m+\gamma_n)z_t-(\gamma_m+\gamma_n)z_{t'}\}}\mathrm{Cov}(s_{\rho_s,z_t}(m_u;n_d),s_{\rho_s,z_{t'}}(m_u;n_d))$$
$$- N_m^{(1)}(z_t)N_n^{(1)}(z_t)N_m^{*(2)}(z_{t'})N_n^{*(1)}(z_{t'})e^{i\Re\{(\gamma_m+\gamma_n)z_t-(-\gamma_m+\gamma_n)z_{t'}\}}\mathrm{Cov}(s_{\rho_s,z_t}(m_u;n_d),s_{\rho_s,z_{t'}}(m_d;n_d))$$
$$- N_m^{(1)}(z_t)N_n^{(1)}(z_t)N_m^{*(1)}(z_{t'})N_n^{*(2)}(z_{t'})e^{i\Re\{(\gamma_m+\gamma_n)z_t-(\gamma_m-\gamma_n)z_{t'}\}}\mathrm{Cov}(s_{\rho_s,z_t}(m_u;n_d),s_{\rho_s,z_{t'}}(m_u;n_u))$$
$$+ N_m^{(1)}(z_t)N_n^{(1)}(z_t)N_m^{*(2)}(z_{t'})N_n^{*(2)}(z_{t'})e^{i\Re\{(\gamma_m+\gamma_n)z_t-(-\gamma_m-\gamma_n)z_{t'}\}}\mathrm{Cov}(s_{\rho_s,z_t}(m_u;n_d),s_{\rho_s,z_{t'}}(m_d;n_u))$$
$$- N_m^{(2)}(z_t)N_n^{(1)}(z_t)N_m^{*(1)}(z_{t'})N_n^{*(1)}(z_{t'})e^{i\Re\{(-\gamma_m+\gamma_n)z_t-(\gamma_m+\gamma_n)z_{t'}\}}\mathrm{Cov}(s_{\rho_s,z_t}(m_d;n_d),s_{\rho_s,z_{t'}}(m_u;n_d))$$
$$+ N_m^{(2)}(z_t)N_n^{(1)}(z_t)N_m^{*(2)}(z_{t'})N_n^{*(1)}(z_{t'})e^{i\Re\{(-\gamma_m+\gamma_n)z_t-(-\gamma_m+\gamma_n)z_{t'}\}}\mathrm{Cov}(s_{\rho_s,z_t}(m_d;n_d),s_{\rho_s,z_{t'}}(m_d;n_d))$$
$$+ N_m^{(2)}(z_t)N_n^{(1)}(z_t)N_m^{*(1)}(z_{t'})N_n^{*(2)}(z_{t'})e^{i\Re\{(-\gamma_m+\gamma_n)z_t-(\gamma_m-\gamma_n)z_{t'}\}}\mathrm{Cov}(s_{\rho_s,z_t}(m_d;n_d),s_{\rho_s,z_t}(m_u;n_u))$$
$$- N_m^{(2)}(z_t)N_n^{(1)}(z_t)N_m^{*(2)}(z_{t'})N_n^{*(2)}(z_{t'})e^{i\Re\{(-\gamma_m+\gamma_n)z_t-(-\gamma_m-\gamma_n)z_{t'}\}}\mathrm{Cov}(s_{\rho_s,z_t}(m_d;n_d),s_{\rho_s,z_{t'}}(m_d;n_u))$$
$$- N_m^{(1)}(z_t)N_n^{(2)}(z_t)N_m^{*(1)}(z_{t'})N_n^{*(1)}(z_{t'})e^{i\Re\{(\gamma_m-\gamma_n)z_t-(\gamma_m+\gamma_n)z_{t'}\}}\mathrm{Cov}(s_{\rho_s,z_t}(m_u;n_u),s_{\rho_s,z_{t'}}(m_u;n_d))$$
$$+ N_m^{(1)}(z_t)N_n^{(2)}(z_t)N_m^{*(2)}(z_{t'})N_n^{*(1)}(z_{t'})e^{i\Re\{(\gamma_m-\gamma_n)z_t-(-\gamma_m+\gamma_n)z_{t'}\}}\mathrm{Cov}(s_{\rho_s,z_t}(m_u;n_u),s_{\rho_s,z_{t'}}(m_d;n_d))$$
$$+ N_m^{(1)}(z_t)N_n^{(2)}(z_t)N_m^{*(1)}(z_{t'})N_n^{*(2)}(z_{t'})e^{i\Re\{(\gamma_m-\gamma_n)z_t-(\gamma_m-\gamma_n)z_{t'}\}}\mathrm{Cov}(s_{\rho_s,z_t}(m_u;n_u),s_{\rho_s,z_{t'}}(m_u;n_u))$$
$$- N_m^{(1)}(z_t)N_n^{(2)}(z_t)N_m^{*(2)}(z_{t'})N_n^{*(2)}(z_{t'})e^{i\Re\{(\gamma_m-\gamma_n)z_t-(-\gamma_m-\gamma_n)z_{t'}\}}\mathrm{Cov}(s_{\rho_s,z_t}(m_u;n_u),s_{\rho_s,z_{t'}}(m_d;n_u))$$

$$+ N_m^{(2)}(z_t)N_n^{(2)}(z_t)N_m^{*(1)}(z_{t'})N_n^{*(1)}(z_{t'})e^{i\Re\{(-\gamma_m-\gamma_n)z_t-(\gamma_m+\gamma_n)z_{t'}\}}\mathrm{Cov}(s_{\rho_s,z_t}(m_\mathrm{d};n_\mathrm{u}),s_{\rho_s,z_{t'}}(m_\mathrm{u};n_\mathrm{d}))$$
$$- N_m^{(2)}(z_t)N_n^{(2)}(z_t)N_m^{*(2)}(z_{t'})N_n^{*(1)}(z_{t'})e^{i\Re\{(-\gamma_m-\gamma_n)z_t-(-\gamma_m+\gamma_n)z_{t'}\}}\mathrm{Cov}(s_{\rho_s,z_t}(m_\mathrm{d};n_\mathrm{u}),s_{\rho_s,z_{t'}}(m_\mathrm{d};n_\mathrm{d}))$$
$$- N_m^{(2)}(z_t)N_n^{(2)}(z_t)N_m^{*(1)}(z_{t'})N_n^{*(2)}(z_{t'})e^{i\Re\{(-\gamma_m-\gamma_n)z_t-(\gamma_m-\gamma_n)z_{t'}\}}\mathrm{Cov}(s_{\rho_s,z_t}(m_\mathrm{d};n_\mathrm{u}),s_{\rho_s,z_{t'}}(m_\mathrm{u};n_\mathrm{u}))$$
$$+ N_m^{(2)}(z_t)N_n^{(2)}(z_t)N_m^{*(2)}(z_{t'})N_n^{*(2)}(z_{t'})e^{i\Re\{(-\gamma_m-\gamma_n)z_t-(-\gamma_m-\gamma_n)z_{t'}\}}\mathrm{Cov}(s_{\rho_s,z_t}(m_\mathrm{d};n_\mathrm{u}),s_{\rho_s,z_{t}}(m_\mathrm{d};n_\mathrm{u}))]$$
$$\times e^{-\Im\{(\gamma_m+\gamma_n)(z_t+z_{t'})\}}, \qquad (72)$$

depends on the covariances of the scatter function density in the forward azimuth that couple incident mode n to scattered mode m.

When Eq. (71) is expressed in terms of the depth-integrated second moment of the incident field in the absence of the shell using Eq. (32), it is found that

$$\int_0^\infty \frac{1}{d(z)}\langle|\Phi_s(\mathbf{r}|\mathbf{r_0},\Delta\rho_s(\rho_s))|^2\rangle \mathrm{d}z = \sum_{n=1}^{\infty} W_i^{(n)}(\boldsymbol{\rho}|\mathbf{r_0})\mu_n^{\mathrm{cor}}\,\Delta\rho_s, \qquad (73)$$

which is the desired form of general Eq. (25), where

$$\mu_n^{\mathrm{cor}}(\rho_s) = \sum_m \frac{1}{|\xi_m|}\int_0^\infty \mathrm{d}z_t\int_0^\infty \mathrm{d}z_{t'}\frac{l_x(\rho_s,z_t,z_{t'})}{\xi_m} \times\frac{4\pi^2}{k(z_t)k(z_{t'})d(z_t)d(z_{t'})}C_{s,s}(\rho_s,z_t,z_{t'},m,n) \qquad (74)$$

is the *exponential coefficient of modal field variance*. It contains a modal sum that accounts for coupling between the nth mode and every other mode in the waveguide due to the *random* scattering process. If the scattering process is not random, the covariance of the scatter function density is zero and, consequently, μ_n is also zero.

Attenuation due to deterministic absorption in the medium is already included in the incident field through $W_i^{(n)}$ of Eq. (32) in Eq. (73). The effect of scattering must be determined separately to conserve energy in the current marching formulation. This requires $-2\Im\{\xi_n-\xi_m\}$ to be set to zero in going from Eq. (71) to Eqs. (73) and (74), following an approach similar to that described in deriving the waveguide extinction theorem.[6] An equivalent route with much historical precedence would be to derive the total field moments without absorption and then include it in the incident field at the final stage.

It is noteworthy that μ_n may depend on shell range ρ_s since it describes the potentially range-dependent variations of the medium's inhomogeneities. When the inhomogeneities obey a stationary random process in range, μ_n is a constant, independent of shell range ρ_s.

2. Uncorrelated scatterers within the Fresnel width

Here inhomogeneities within the shell at range ρ_s satisfy $l_y < Y_\mathrm{F}(\rho,\rho_s)$ or $|\rho_s-\rho/2| < (\rho/2)\sqrt{1-4l_y^2/\lambda\rho}$. Their scatter function densities then decorrelate in both range and cross-range within the Fresnel width. For scatter function density separations greater than that which can fall within the coherence area $A_c(\rho_s,z_t,z_{t'})$, defined in Appendix A, scatter function densities are assumed to be uncorrelated, so that

$$\langle s_{\mathbf{r_t}}(\pi-\alpha_m,\beta_s(\phi,\phi_t);\alpha_n,\phi_t)s^*_{\mathbf{r_{t'}}}(\pi-\alpha_{m'},\beta_s(\phi,\phi_{t'});\alpha_{n'},\phi_{t'})\rangle$$
$$\approx A_c(\rho_s,z_t,z_t')[\langle s_{\rho_s,z_t}(\pi-\alpha_m,\beta_s(\phi,\phi_t);\alpha_n,\phi_t)s^*_{\rho_s,z_{t'}}(\pi-\alpha_{m'},\beta_s(\phi,\phi_t);\alpha_{n'},\phi_t)\rangle - \langle s_{\rho_s,z_t}(\pi-\alpha_m,\beta_s(\phi,\phi_t);\alpha_n,\phi_t)\rangle$$
$$\times\langle s^*_{\rho_s,z_{t'}}(\pi-\alpha_{m'},\beta_s(\phi,\phi_t);\alpha_{n'},\phi_t)\rangle]\delta(\boldsymbol{\rho}_t-\boldsymbol{\rho}_{\mathbf{t'}}) + \langle s_{\rho_s,z_t}(\pi-\alpha_m,\beta_s(\phi,\phi_t);\alpha_n,\phi_t)\rangle\langle s^*_{\rho_s,z_{t'}}(\pi-\alpha_{m'},\beta_s(\phi,\phi_{t'});\alpha_{n'},\phi_{t'})\rangle. \qquad (75)$$

Substituting Eq. (75) into Eq. (61), leads to the second moment of the direct wave portion of the scattered field from the shell, which has 16 terms of similar form. The first of these is

$$\langle|\Phi_s(\mathbf{r}|\mathbf{r_0},\Delta\rho_s(\rho_s))|^2\rangle_1$$
$$= \sum_m\sum_n\sum_{m'}\sum_{n'}\int_0^\infty \mathrm{d}z_t\int_0^\infty \mathrm{d}z_{t'}\int_{\rho_s-\Delta\rho_s/2}^{\rho_s+\Delta\rho_s/2}\mathrm{d}\rho_t\int_{\phi-\phi_F/2}^{\phi+\phi_F/2}\rho_t\,\mathrm{d}\phi_t\, e^{i(\xi_m-\xi_{m'})[\rho\rho_t/2(\rho-\rho_t)](\phi_t-\phi)^2}$$
$$\times A_c(\rho_s,z_t,z_{t'})\frac{4\pi^2}{k(z_t)k(z_{t'})d(z_t)d(z_{t'})}\frac{1}{d^2(z_0)}\frac{1}{\sqrt{\xi_m\xi^*_{m'}}}\frac{1}{\sqrt{\xi_n\xi^*_{n'}}}\frac{1}{(\rho-\rho_t)\rho_t}$$
$$\times u_m(z)u^*_{m'}(z)u_n(z_0)u^*_{n'}(z_0)N_m^{(1)}(z_t)N_n^{(1)}(z_t)N_{m'}^{*(1)}(z_{t'})N_{n'}^{*(1)}(z_{t'})$$

$$\times[\langle s_{\rho_s,z_t}(\pi-\alpha_m,\beta_s(\phi,\phi_t);\alpha_n,\phi_t)s^*_{\rho_s,z_{t'}}(\pi-\alpha_{m'},\beta_s(\phi,\phi_t);\alpha_{n'},\phi_t)\rangle$$
$$-\langle s_{\rho_s,z_t}(\pi-\alpha_m,\beta_s(\phi,\phi_t);\alpha_n,\phi_t)\rangle\langle s^*_{\rho_s,z_{t'}}(\pi-\alpha_{m'},\beta_s(\phi,\phi_t);\alpha_{n'},\phi_t)\rangle]$$
$$\times\, e^{i\Re\{\xi_m-\xi_{m'}\}(\rho-\rho_t)}e^{i\Re\{\xi_n-\xi_{n'}\}\rho_t}e^{-\Im\{\xi_m+\xi_{m'}\}(\rho-\rho_t)}e^{-\Im\{\xi_n+\xi_{n'}\}\rho_t}$$
$$\times\, e^{i\Re\{(\gamma_m+\gamma_n)z_t-(\gamma_{m'}+\gamma_{n'})z_{t'}\}}e^{-\Im\{(\gamma_m+\gamma_n)z_t+(\gamma_{m'}+\gamma_{n'})z_{t'}\}}$$
$$+|\langle\Phi_s(\mathbf{r}|\mathbf{r_0},\Delta\rho_s(\rho_s))\rangle_1|^2. \tag{76}$$

Here $|\langle\Phi_s(\mathbf{r}|\mathbf{r_0},\Delta\rho_s(\rho_s))\rangle_1|^2$ is the first term of the square of the mean scattered field from the shell, from Eq. (13). It is negligible, as noted in previous sections.

Following on analysis analogous to that in Sec. IV B 1, the exponential coefficient of modal field variance is found to be

$$\mu_n^{\text{uncor}}(\rho_s)=\sum_m\sqrt{\frac{\rho}{2\pi\xi_m\rho_s(\rho-\rho_s)}}\frac{1}{|\xi_m|}\times\int_0^\infty \mathrm{d}z_t\int_0^\infty \mathrm{d}z_{t'}\,A_c(\rho_s,z_t,z_{t'})\times\frac{4\pi^2}{k(z_t)k(z_{t'})d(z_t)d(z_{t'})}C_{s,s}(\rho_s,z_t,z_{t'},m,n), \tag{77}$$

which strongly depends on shell range ρ_s.

C. Coherent second moment interference between incident and scattered fields

We now derive Eq. (26) for the depth-integrated cross terms between the incident and scattered field from the isolated shell. One of the two cross-terms can be approximated as

$$\langle\Phi(\mathbf{r}|\mathbf{r_0})\Phi_s^*(\mathbf{r}|\mathbf{r_0},\Delta\rho_s(\rho_s))\rangle=\langle\Phi_i(\mathbf{r}|\mathbf{r_0})\Phi_s^*(\mathbf{r}|\mathbf{r_0},\Delta\rho_s(\rho_s))\rangle=\sum_n\Phi_i^{(n)}(\mathbf{r}|\mathbf{r_0})\sum_m\Phi_i^{*(m)}(\mathbf{r}|\mathbf{r_0})(-i\nu_m^*)\Delta\rho_s \tag{78}$$

$$=\sum_n\sum_m\frac{2\pi}{d^2(z_0)}u_n(z)u_m^*(z)u_n(z_0)u_m^*(z_0)\frac{e^{i\Re\{\xi_n-\xi_m\}\rho}}{\rho\sqrt{\xi_n\xi_m^*}}\times e^{-\Im\{\xi_n+\xi_m\}\rho}(-i\nu_m^*)\Delta\rho_s, \tag{79}$$

by application of Eq. (59) with the same single scatter approximation as before. A similar expression can be obtained for the other cross-term.

Integrating Eq. (79) over the receiver depth, invoking modal orthogonality Eq. (47), and applying Eq. (32) leads to

$$\int_0^\infty\langle\Phi_i(\mathbf{r}|\mathbf{r_0})\Phi_s^*(\mathbf{r}|\mathbf{r_0},\Delta\rho_s(\rho_s))\rangle\mathrm{d}z=\sum_n W_i^{(n)}(\boldsymbol{\rho}|\mathbf{r_0})(-i\nu_n^*(\rho_s))\Delta\rho_s. \tag{80}$$

Similarly, it can be shown that

$$\int_0^\infty\langle\Phi_i^*(\mathbf{r}|\mathbf{r_0})\Phi_s(\mathbf{r}|\mathbf{r_0},\Delta\rho_s(\rho_s))\rangle\mathrm{d}z=\sum_n W_i^{(n)}(\boldsymbol{\rho}|\mathbf{r_0})(i\nu_n(\rho_s))\Delta\rho_s. \tag{81}$$

Summing Eqs. (80) and (81), we find

$$\int_0^\infty[\langle\Phi_i(\mathbf{r}|\mathbf{r_0})\Phi_s^*(\mathbf{r}|\mathbf{r_0},\Delta\rho_s(\rho_s))\rangle+\langle\Phi_i^*(\mathbf{r}|\mathbf{r_0})\Phi_s(\mathbf{r}|\mathbf{r_0},\Delta\rho_s(\rho_s))\rangle]\mathrm{d}z=-\sum_n W_i^{(n)}(\boldsymbol{\rho}|\mathbf{r_0})2\Im\{\nu_n(\rho_s)\}\Delta\rho_s, \tag{82}$$

which depends on the modal attenuation coefficient $\Im\{\nu_n\}$. This is a consequence of the extinction or forward scatter theorem that states that it is the coherent interference between the incident and scattered fields that leads to shadow formation and the eventual attenuation of the forward field.

The modal power equation (28) and the corresponding difference equation (29) for scattering from a single shell follow directly from the derivation for Eq. (26) and the fact that the depth-integrated intensity at the receiver in the absence of inhomogeneities is described by Eq. (32).

V. SOLUTIONS FOR THE MEAN, VARIANCE, INTENSITY, POWER, AND SIGNAL-TO-NOISE RATIO OF THE FORWARD FIELD

Within the framework of the present formulation, we now provide general solutions for the mean, second moment, variance, and power of the forward field in terms of the parameters needed to describe the incident field and the first two statistical moments of the random medium's scatter function density, which obeys a spatial random process that need not be stationary. These solutions include the accumulated effects of multiple forward scatter from source to receiver through range integrals of the horizontal wave number change ν_n and the coefficient of field variance μ_n for the nth mode. We then evaluate these integrals analytically for the special case when the inhomogeneities obey a stationary random process along the forward propagation path from source to receiver.

A. Field moments at a single receiver, power, and signal-to-noise ratio

1. Mean forward field

The mean of the forward field received at $\mathbf{r}$ from a source at $\mathbf{r_0}$ after propagation through a random inhomogeneous waveguide can be expressed as a sum of modal contributions,

$$\langle \Psi_T(\mathbf{r}|\mathbf{r_0})\rangle = \sum_{n=1}^{M_{max}} 4\pi \frac{i}{d(z_0)\sqrt{8\pi}} e^{-i\pi/4} u_n(z) u_n(z_0) \times \frac{e^{i\xi_n\rho}}{\sqrt{\xi_n\rho}} e^{i\int_0^{\rho}\nu_n(\rho_s)d\rho_s}, \quad (83)$$

from Eqs. (18) and (22). Here the wave number change $\nu_n(\rho_s)$ defined in Eq. (60) depends on the expected scattering properties of the random medium, which may be spatially nonstationary.

2. Variance of the forward field

The variance of the forward field received at $\mathbf{r}$ from the source at $\mathbf{r_0}$ can be expressed as a sum of modal variance terms,

$$\mathrm{Var}(\Psi_T(\mathbf{r}|\mathbf{r_0})) = \sum_{n=1}^{M_{max}} \frac{2\pi}{d^2(z_0)} \frac{1}{|\xi_n|\rho} |u_n(z_0)|^2 |u_n(z)|^2 \times e^{-2\mathfrak{I}\left\{\xi_n\rho + \int_0^{\rho}\nu_n(\rho_s)d\rho_s\right\}}\left(e^{\int_0^{\rho}\mu_n(\rho_s)d\rho_s} - 1\right), \quad (84)$$

from Eq. (49), with the use of Eq. (32). For each mode n, the forward field variance depends on the expected modal extinction cross section of medium inhomogeneities through $\mathfrak{I}\{\nu_n\}$, and the covariance of the scatter function density of these inhomogeneities through μ_n. Both coefficients account for coupling of each mode n to every other mode in the random waveguide. Both coefficients may vary with range to account for nonstationarity in the inhomogeneous medium and variation of the medium's cross-range coherence length with respect to the local Fresnel width. The exponential coefficient of field variance $\mu_n(\rho_s)$ is given in Eq. (74) for the case when scatterers are correlated, and in Eq. (77) uncorrelated, within the local Fresnel width. The integral of $\mu_n(\rho_s)$ over range ρ_s is evaluated in Sec. V B for the important special case, where the medium's inhomogeneities obey a stationary random process between the source and the receiver.

This theory is consistent with the fact that in the limiting case of a nonrandom waveguide, the variance of the forward field must be zero. For example, in a waveguide with a static and uniform distribution of nonrandom inhomogeneities, the covariance of the scatter function density is zero. This then makes the exponential coefficient of field variance $\mu_n(\rho_s)$ zero for each mode from Eq. (74) or (77). The variance of the forward field in Eq. (84) is then zero, as it should be for a completely deterministic and coherent field.

3. Second moment of the forward field

The second moment of the forward field received at $\mathbf{r}$ from a source at $\mathbf{r_0}$ in the random inhomogeneous waveguide is found to be

$$\langle|\Psi_T(\mathbf{r}|\mathbf{r}_0)|^2\rangle = \sum_{n=1}^{M_{\max}} \frac{2\pi}{d^2(z_0)} \frac{1}{|\xi_n|\rho} |u_n(z_0)|^2 |u_n(z)|^2 e^{-2\mathfrak{I}\left\{\xi_n\rho + \int_0^{\rho}\nu_n(\rho_s)d\rho_s\right\}}\left(e^{\int_0^{\rho}\mu_n(\rho_s)d\rho_s} - 1\right) + \sum_{n=1}^{M_{\max}}\sum_{m=1}^{M_{\max}} \frac{2\pi}{d^2(z_0)} \frac{1}{\sqrt{\xi_n\xi_m^*}\rho} u_n(z_0)u_m^*(z_0)u_n(z)u_m^*(z) e^{i\mathfrak{R}\left\{(\xi_n-\xi_m)\rho + \int_0^{\rho}(\nu_n(\rho_s)-\nu_m(\rho_s))d\rho_s\right\}} \times e^{-\mathfrak{I}\left\{(\xi_n+\xi_m)\rho + \int_0^{\rho}(\nu_n(\rho_s)+\nu_m(\rho_s))d\rho_s\right\}} \quad (85)$$

by inserting Eqs. (36), (37), and (84) into Eq. (42). The first term in Eq. (85) corresponds to the field variance and the second to the mean field squared. When the mean field dominates, the expected total intensity will fluctuate as a function of range due to coherent interference between waveguide modes. This interference is maintained even in the presence of inhomogeneities but may be significantly different from that of the incident field due to dispersion induced by the inhomogeneities. When the variance dominates, the modal interference patterns become insignificant, and the expected total intensity decays monotonically as the range becomes large.[16]

4. Forward field power

The second moment of the forward field integrated over the receiver depth is directly proportional to the net power propagated to range ρ in the waveguide. With the use of Eqs. (32) and (34), it can be expressed as

$$\langle W_T(\mathbf{r}|\mathbf{r_0})\rangle = \sum_{n=1}^{M_{\max}} \frac{2\pi}{d^2(z_0)} \frac{1}{|\xi_n|\rho} |u_n(z_0)|^2 e^{-2\Im\{\xi_n\rho+\int_0^\rho \nu_n(\rho_s)d\rho_s\}} \times e^{\int_0^\rho \mu_n(\rho_s)d\rho_s}, \tag{86}$$

which decays both monotonically with range, by the waveguide extinction theorem,[6] and uniformly since depth integration eliminates modal interference. This decay reflects the total power loss in the waveguide due to both scattering and absorption in the present formulation.

5. Signal to noise ratio

Here we define the signal to noise ratio $\mathrm{SNR}_n(\rho)$ for the nth mode of the forward field as the ratio of the depth-integrated square of the mean forward field to the depth-integrated variance. It is found to be

$$\mathrm{SNR}_n(\rho) = 1/\left(e^{\int_0^\rho \mu_n(\rho_s)d\rho_s} - 1\right), \tag{87}$$

by integrating Eqs. (83) and (84) over receiver depth. We observe that the signal to noise ratio for each mode only depends on the coefficient of modal field variance μ_n, is independent of the wave number change ν_n, and decreases with increasing receiver range ρ.

B. Special solutions when inhomogeneities obey a stationary random process along the path from source to receiver

The mean, second moment, and variance of the forward field can be readily obtained from Eqs. (83), (84), and (85), respectively, if statistical properties of the medium's inhomogeneities are known. The horizontal wave number change ν_n and the coefficient of modal field variance μ_n can then be determined as a function of range from source to receiver for each mode n.

Here we will assume that the scatter function density of the inhomogeneities follows the same stationary random process across all active regions from source to receiver. We can then analytically integrate ν_n and μ_n across range to obtain closed form expressions for the field moments. In this case, from Eq. (60), ν_n is simply a constant independent of shell range ρ_s so that $\int_0^\rho \nu_n \, d\rho_s = \nu_n\rho$. Since μ_n depends on the relative size of the cross-range coherence length l_y to the Fresnel width at the given shell range, however, three forms of solution are possible and must be considered.

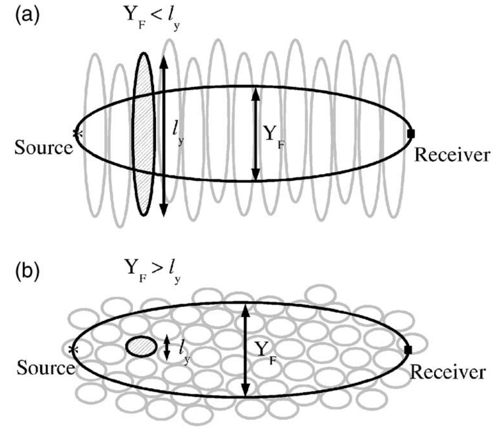

FIG. 1. (a) The maximum Fresnel width $Y_F(\rho,\rho/2)$ is smaller than the cross-range coherence length l_y of the local inhomogeneities for the given source-receiver separation ρ. (b) The maximum Fresnel width $Y_F(\rho,\rho/2)$ is larger than the cross-range coherence length l_y of the local inhomogeneities for the given source receiver separation ρ. Local inhomogeneities are fully correlated in cross-range within the Fresnel width for all single scatter shells in (a), while in (b) the active region within the Fresnel width contains inhomogeneities uncorrelated in cross-range for single scatter shells located at ρ_s that satisfy $\rho-\rho_s^{\mathrm{cor}}<\rho_s<\rho+\rho_s^{\mathrm{cor}}$. Figure not to scale.

1. The Fresnel width exceeds the cross-range coherence length of the scatter function density over part of the propagation path

This is the most general situation. The Fresnel width exceeds the cross-range coherence length l_y of the medium's scatter function density over the middle segment ($\rho_s^{\mathrm{cor}}<\rho_s<\rho-\rho_s^{\mathrm{cor}}$) of the propagation path, but not at the beginning ($\rho_s\le\rho_s^{\mathrm{cor}}$) or end ($\rho_s>\rho-\rho_s^{\mathrm{cor}}$), where $\rho_s^{\mathrm{cor}}=\rho/2(1-\sqrt{1-4l_y^2/\lambda\rho})$. Cross-range decorrelation of the scatter function density then only occurs over the middle segment. The exponential coefficient for field variance then takes on a different form for each of these segments in the marching solution from source to receiver,

$$\int_0^\rho \mu_n(\rho_s)d\rho_s = \int_0^{\rho_s^{\mathrm{cor}}} \mu_n^{\mathrm{cor}}(\rho_s)d\rho_s + \int_{\rho_s^{\mathrm{cor}}}^{\rho-\rho_s^{\mathrm{cor}}} \mu_n^{\mathrm{uncor}}(\rho_s)d\rho_s + \int_{\rho-\rho_s^{\mathrm{cor}}}^{\rho} \mu_n^{\mathrm{cor}}(\rho_s)d\rho_s. \tag{88}$$

If the coherence scales, l_y and A_c are independent of depth, substituting Eqs. (74) and (77) for $\mu_n^{\mathrm{cor}}(\rho_s)$ and $\mu_n^{\mathrm{uncor}}(\rho_s)$, respectively, into Eq. (88), leads to

$$\int_0^\rho \mu_n(\rho_s)d\rho_s = \sum_m \left(\frac{l_x}{\xi_m}\int_0^{\rho_s^{\mathrm{cor}}} d\rho_s + A_c\int_{\rho_s^{\mathrm{cor}}}^{\rho-\rho_s^{\mathrm{cor}}} \sqrt{\frac{\rho}{2\pi\xi_m\rho_s(\rho-\rho_s)}}d\rho_s + \frac{l_x}{\xi_m}\int_{\rho-\rho_s^{\mathrm{cor}}}^{\rho} d\rho_s\right)\frac{1}{|\xi_m|} \times \int_0^\infty dz_t \int_0^\infty dz_{t'} \frac{4\pi^2}{k(z_t)k(z_{t'})d(z_t)d(z_{t'})} C_{s,s}(z_t,z_{t'},m,n). \tag{89}$$

Evaluting the integrals over the three segments then leads to the solution

$$\int_0^\rho \mu_n(\rho_s)\mathrm{d}\rho_s = \sum_m 2\left(\frac{l_x}{\xi_m}\rho_s^{\mathrm{cor}} + A_c\sqrt{\frac{\rho}{2\pi\xi_m}}\left[\sin^{-1}\sqrt{1-\frac{\rho_s^{\mathrm{cor}}}{\rho}} - \sin^{-1}\sqrt{\frac{\rho_s^{\mathrm{cor}}}{\rho}}\right]\right)\frac{1}{|\xi_m|} \times \int_0^\infty \mathrm{d}z_t \int_0^\infty \mathrm{d}z_{t'} \frac{4\pi^2}{k(z_t)k(z_{t'})d(z_t)d(z_{t'})} C_{s,s}(z_t,z_{t'},m,n). \quad (90)$$

2. The cross-range coherence length of the scatter function density exceeds the Fresnel width over the entire propagation path

Here the scatterers are assumed to be fully correlated in cross-range across the entire propagation path. This is valid when $l_y > Y_{\mathrm{F}}(\rho,\rho/2)$ or $\rho < 4l_y^2/\lambda$, since the maximum Fresnel width falls midway between the source and the receiver. This case is illustrated in Fig. 1(a). Integrating Eq. (74) leads to the solution

$$\int_0^\rho \mu_n(\rho_s)\mathrm{d}\rho_s = \int_0^\rho \mu_n^{\mathrm{cor}}(\rho_s)\mathrm{d}\rho_s = \mu_n^{\mathrm{cor}}\rho, \quad (91)$$

since μ_n^{cor} is range independent here.

The signal to noise ratio for each incident mode n is then approximately

$$\mathrm{SNR}_n(\rho) = 1/(\mathrm{e}^{\mu_n^{\mathrm{cor}}\rho} - 1), \quad (92)$$

which becomes $\mathrm{SNR}_n(\rho) \approx 1/\mu_n^{\mathrm{cor}}\rho$ for sufficiently small ranges and $\mathrm{SNR}_n(\rho) \approx \mathrm{e}^{-\mu_n^{\mathrm{cor}}\rho}$ for sufficiently large ranges as long as Eq. (92) remains valid.

3. Fresnel length exceeds the cross-range coherence length and waveguide depth exceeds the vertical coherence length over the entire propagation path

Here it is assumed that scatterers may be uncorrelated in all three dimensions within the active region of any shell along the propagation path. This case is illustrated in Fig. 1(b). This is particularly applicable to propagation through bubble clouds, schools of fish, or fine-scale turbulence in the ocean. It can be described by taking the scatter function densities at $\mathbf{r_t}$ and $\mathbf{r_{t'}}$ to be uncorrelated when their separation is greater than that which can fall within a coherence volume $V_c(z_t)$, so that

$$\begin{aligned}\langle s_{\mathbf{r_t}}(\pi-\alpha_m,\beta_s(\phi,\phi_t);\alpha_n,\phi_t)s^*_{\mathbf{r_{t'}}}(\pi-\alpha_{m'},\beta_s(\phi,\phi_{t'});\alpha_{n'},\phi_{t'})\rangle \\ = V_c(z_t)[\langle s_{z_t}(\pi-\alpha_m,\beta_s(\phi,\phi_t);\alpha_n,\phi_t)s^*_{z_t}(\pi-\alpha_{m'},\beta_s(\phi,\phi_t);\alpha_{n'},\phi_t)\rangle \\ - \langle s_{z_t}(\pi-\alpha_m,\beta_s(\phi,\phi_t);\alpha_n,\phi_t)\rangle\langle s^*_{z_t}(\pi-\alpha_{m'},\beta_s(\phi,\phi_t);\alpha_{n'},\phi_t)\rangle]\delta(\mathbf{r_t}-\mathbf{r_{t'}}) \\ + \langle s_{z_t}(\pi-\alpha_m,\beta_s(\phi,\phi_t);\alpha_n,\phi_t)\rangle\langle s^*_{z_{t'}}(\pi-\alpha_{m'},\beta_s(\phi,\phi_{t'});\alpha_{n'},\phi_{t'})\rangle.\end{aligned} \quad (93)$$

Following the analysis of Sec. IV B, but instead using Eq. (93) for the correlation of the scatter function density in Eq. (61), the coefficient of modal field variance becomes

$$\mu_n^{\mathrm{uncor,3\text{-}D}}(\rho_s) = \sum_m \sqrt{\frac{\rho}{2\pi\xi_m\rho_s(\rho-\rho_s)}}\frac{1}{|\xi_m|}\int_0^\infty \mathrm{d}z_t\, V_c(z_t)\frac{4\pi^2}{k^2(z_t)d^2(z_t)}C_{s,s}(z_t,z_t,m,n). \quad (94a)$$

Integrating Eq. (94a) leads to the solution

$$\int_0^\rho \mu_n(\rho_s)\mathrm{d}\rho_s = \int_0^\rho \mu_n^{\mathrm{uncor,3\text{-}D}}(\rho_s)\mathrm{d}\rho_s = \sum_m \sqrt{\frac{\rho\pi}{2\xi_m}}\frac{1}{|\xi_m|}\int_0^\infty \mathrm{d}z_t\, V_c(z_t)\frac{4\pi^2}{k^2(z_t)d^2(z_t)}C_{s,s}(z_t,z_t,m,n). \quad (94b)$$

The signal to noise ratio for each incident mode n is

$$\mathrm{SNR}_n(\rho) = 1/(\mathrm{e}^{\mathcal{M}_n^{\mathrm{uncor,3\text{-}D}}\sqrt{\rho}} - 1), \quad (95)$$

where

$$\mathcal{M}_n^{\mathrm{uncor,3\text{-}D}} = \sum_m \sqrt{\frac{\pi}{2\xi_m}}\frac{1}{|\xi_m|}\int_0^\infty \mathrm{d}z_t\, V_c(z_t)\frac{4\pi^2}{k^2(z_t)d^2(z_t)}C_{s,s}(z_t,z_t,m,n). \quad (96)$$

The signal to noise ratio becomes $\mathrm{SNR}_n(\rho) \approx 1/\mathcal{M}_n^{\text{uncor,3-D}}\sqrt{\rho}$ for sufficiently small ranges, and $\mathrm{SNR}_n(\rho) \approx \mathrm{e}^{-\mathcal{M}_n^{\text{uncor,3-D}}\sqrt{\rho}}$ for sufficiently large ranges as long as Eq. (95) is valid.

By comparing Eqs. (92) and (95), it is observed that the direct wave for each mode is more coherent when decorrelation occurs in depth and azimuth as well as in range within the active region. This analytically shows that 3-D scattering effects become important as the Fresnel width exceeds the medium's local cross-range coherence length.

In the present scenario of three-dimensional decorrelation within the active region, the modal attenuation coefficient, from Eq. (60), can be expressed as

$$\mathfrak{I}\{\nu_n\} = \int_0^\infty \frac{1}{2}|u_n(z_t)|^2 \frac{1}{d(z_t)} \frac{1}{V_c(z_t)} \langle \sigma_n(0,z_t) \rangle dz_t. \tag{97}$$

This follows from Eq. (20) of Ref. 6 for the modal extinction cross-section σ_n of an inhomogeneity and Eq. (A5), which relates its scatter function to a corresponding scatter function density. Equation (97) is convenient because it can be used to calculate the attenuation due to scattering of each mode.

VI. MUTUAL INTENSITY AND THE SPATIAL COVARIANCE OF THE FORWARD FIELD

Here we provide analytic expressions for the mutual intensity and covariance of the forward field for two receivers at $\mathbf{r_1} = (\rho_1, \phi_1, z_1)$ and $\mathbf{r_2} = (\rho_2, \phi_2, z_2)$ in a random inhomogeneous ocean waveguide. This analysis is applicable to any two receivers in an arbitrary array configuration.

A. Difference and integral equations

Following an analysis similar to that in Sec. III B, it can be shown that by considering scattering from an elemental cylindrical shell of inhomogeneities, the change in the depth-integrated cross-moment, or the depth-integrated mutual intensity, of the forward field $\Delta W^{(n)}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2 | \mathbf{r}_0)$ at ρ_1 and ρ_2 for each mode m satisfies

$$\begin{aligned}\Delta\langle W^{(n)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2|\mathbf{r}_0)\rangle = \langle W^{(n)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2|\mathbf{r}_0)\rangle(&\mu_{n,1,2}(\rho_s) \\ &+ i\Re\{\nu_{n,1}(\rho_s) - \nu_{n,2}(\rho_s)\} \\ &- \mathfrak{I}\{\nu_{n,1}(\rho_s) + \nu_{n,2}(\rho_s)\})\Delta\rho_s.\end{aligned} \tag{98}$$

Here $\langle W^{(n)}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2 | \mathbf{r}_0)\rangle$ is defined in

$$\int_0^\infty \frac{1}{d(z)} \langle \Phi(\mathbf{r}_1|\mathbf{r}_0)\Phi^*(\mathbf{r}_2|\mathbf{r}_0)\rangle \mathrm{d}z = \sum_n \langle W^{(n)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2|\mathbf{r}_0)\rangle, \tag{99}$$

and $\nu_{n,1}$ and $\nu_{n,2}$ are the modal horizontal wave number changes given in Eq. (60) with scatter functions evaluated at the forward azimuths $\phi = \phi_1$ and $\phi = \phi_2$, respectively, and $\mu_{n,1,2}$ is the exponential coefficient of modal field *covariance*, which will be derived in Sec. VI C.

If the receivers are within a single-scatter shell width in horizontal range, their depth-integrated mutual coherence can be approximated by marching Eq. (98) to a range at the midpoint between the two receivers,

$$\begin{aligned}&\int_{W_i^{(n)}}^{W_T^{(n)}} \frac{\mathrm{d}W^{(n)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2|\mathbf{r_0})}{W^{(n)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2|\mathbf{r_0})} \\ &\quad = \int_0^{\rho_1+\Delta\rho/2} (\mu_{n,1,2}(\rho_s) + i\Re\{\nu_{n,1}(\rho_s) - \nu_{n,2}(\rho_s)\} - \mathfrak{I}\{\nu_{n,1}(\rho_s) + \nu_{n,2}(\rho_s)\})\mathrm{d}\rho_s,\end{aligned} \tag{100}$$

where

$$W_i^{(n)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2|\mathbf{r}_0) = \frac{2\pi}{d^2(z_0)} \frac{1}{|\xi_n|\sqrt{\rho_1\rho_2}} |u_n(z_0)|^2 \mathrm{e}^{i\Re\{\xi_n\}(\rho_1-\rho_2)} \mathrm{e}^{-\mathfrak{I}\{\xi_n\}(\rho_1+\rho_2)} \tag{101}$$

is the depth-integrated mutual intensity of the incident field for mode n at receiver ranges ρ_1 and ρ_2.

The resulting contribution to the depth-integrated mutual intensity of the forward field for the nth mode is then

$$\begin{aligned}&W_T^{(n)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2|\mathbf{r_0}) \\ &\quad = W_i^{(n)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2|\mathbf{r_0})\exp\left(\int_0^{\rho_1+\Delta\rho/2} (\mu_{n,1,2}(\rho_s) + i\Re\{\nu_{n,1}(\rho_s) - \nu_{n,2}(\rho_s)\} - \mathfrak{I}\{\nu_{n,1}(\rho_s) + \nu_{n,2}(\rho_s)\})\mathrm{d}\rho_s\right),\end{aligned} \tag{102}$$

which is valid when $(\mu_{n,1,2} - \mathfrak{I}\{\nu_{n,1} + \nu_{n,2}\})\Delta\rho/2 \ll 1$. This condition is satisfied when the effect of scattering over the horizontal range separation $\Delta\rho$ between the two receivers is small. It is consistent with our basic assumption that the scattered field from any individual single-scatter shell must be small compared to the incident field at the receiver. The condition is only necessary for receivers that are separated in range. It is not required for receivers at different azimuths with the same horizontal range.

B. Solutions

1. Spatial covariance of the forward field

The covariance of the forward fields received at $\mathbf{r_1}$ and $\mathbf{r_2}$ can be expressed as the single modal sum

$$\begin{aligned}
&\mathrm{Cov}(\Psi_T(\mathbf{r_1}|\mathbf{r_0}),\Psi_T(\mathbf{r_2}|\mathbf{r_0})) \\
&= \langle \Psi_T(\mathbf{r_1}|\mathbf{r_0})\Psi_T^*(\mathbf{r_2}|\mathbf{r_0})\rangle - \langle \Psi_T(\mathbf{r_1}|\mathbf{r_0})\rangle\langle \Psi_T^*(\mathbf{r_2}|\mathbf{r_0})\rangle \\
&= \sum_n W_i^{(n)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2|\mathbf{r_0})u_n(z_1)u_n^*(z_2)\exp\left(\int_0^{\rho_1+\Delta\rho/2}(i\Re\{\nu_{n,1}(\rho_s)-\nu_{n,2}(\rho_s)\}-\Im\{\nu_{n,1}(\rho_s)+\nu_{n,2}(\rho_s)\})\mathrm{d}\rho_s\right) \\
&\quad\times\left(\mathrm{e}^{\int_0^{\rho 1+\Delta\rho/2}\mu_{n,1,2}(\rho_s)\mathrm{d}\rho_s}-1\right) \qquad (103)
\end{aligned}$$

$$\begin{aligned}
&= \sum_n \frac{2\pi}{d^2(z_0)}\frac{1}{|\xi_n|\sqrt{\rho_1\rho_2}}|u_n(z_0)|^2u_n(z_1)u_n^*(z_2)\mathrm{e}^{i\Re\{\xi_n\}(\rho_1-\rho_2)}\mathrm{e}^{-\Im\{\xi_n\}(\rho_1+\rho_2)} \\
&\quad\times\exp\left(\int_0^{\rho_1+\Delta\rho/2}(i\Re\{\nu_{n,1}(\rho_s)-\nu_{n,2}(\rho_s)\}-\Im\{\nu_{n,1}(\rho_s)+\nu_{n,2}(\rho_s)\})\mathrm{d}\rho_s\right)\left(\mathrm{e}^{\int_0^{\rho_1+\Delta\rho/2}\mu_{n,1,2}(\rho_s)\mathrm{d}\rho_s}-1\right), \qquad (104)
\end{aligned}$$

using Eqs. (22), (101), and (102).

2. Mutual intensity of forward field

The spatial cross-correlation of the forward fields received at $\mathbf{r_1}$ and $\mathbf{r_2}$ is given by

$$\begin{aligned}
\langle \Psi_T(\mathbf{r_1}|\mathbf{r_0})\Psi_T^*(\mathbf{r_2}|\mathbf{r_0})\rangle &= \sum_n \frac{2\pi}{d^2(z_0)}\frac{1}{|\xi_n|\sqrt{\rho_1\rho_2}}|u_n(z_0)|^2u_n(z_1)u_n^*(z_2)\mathrm{e}^{i\Re\{\xi_n\}(\rho_1-\rho_2)}\mathrm{e}^{-\Im\{\xi_n\}(\rho_1+\rho_2)} \\
&\quad\times\exp\left(\int_0^{\rho_1+\Delta\rho/2}(i\Re\{\nu_{n,1}(\rho_s)-\nu_{n,2}(\rho_s)\}-\Im\{\nu_{n,1}(\rho_s)+\nu_{n,2}(\rho_s)\})\mathrm{d}\rho_s\right)\left(\mathrm{e}^{\int_0^{\rho_1+\Delta\rho/2}\mu_{n,1,2}(\rho_s)\mathrm{d}\rho_s}-1\right) \\
&\quad+\sum_n\sum_m \frac{2\pi}{d^2(z_0)}\frac{1}{\sqrt{\xi_n\xi_m^*\rho_1\rho_2}}u_n(z_0)u_m^*(z_0)u_n(z_1)u_m^*(z_2)\mathrm{e}^{i\Re\{\xi_n\rho_1-\xi_m\rho_2\}}\mathrm{e}^{-\Im\{\xi_n\rho_1+\xi_m\rho_2\}} \\
&\quad\times\exp\left(\int_0^{\rho_1+\Delta\rho/2}(i\Re\{\nu_{n,1}(\rho_s)-\nu_{m,2}(\rho_s)\}-\Im\{\nu_{n,1}(\rho_s)+\nu_{m,2}(\rho_s)\})\mathrm{d}\rho_s,\right) \qquad (105)
\end{aligned}$$

from Eqs. (22) and (104). This spatial cross-correlation is proportional to the mutual intensity of the two receivers.

C. Exponential coefficient of modal covariance

Here we derive an analytic expression for the exponential coefficient of modal covariance $\mu_{m,1,2}$. The spatial cross-correlation of the scattered fields received at $\mathbf{r_1}$ and $\mathbf{r_2}$ from an elemental shell of inhomogeneities obtained from Eq. (9) is

$$\begin{aligned}
&\langle \Phi_s(\mathbf{r_1}|\mathbf{r_0},\Delta\rho_s(\rho_s))\Phi_s^*(\mathbf{r_2}|\mathbf{r_0},\Delta\rho_s(\rho_s))\rangle \\
&\approx \Bigg\langle \sum_m\sum_n\sum_{m'}\sum_{n'}\int_0^\infty \mathrm{d}z_t\int_0^\infty \mathrm{d}z_{t'}\int_{\rho_s-\Delta\rho_s/2}^{\rho_s+\Delta\rho_s/2}\mathrm{d}\rho_t\int_{\rho_s-\Delta\rho_s/2}^{\rho_s+\Delta\rho_s/2}\mathrm{d}\rho_{t'} \\
&\times\int_{\phi_1-\phi_F/2}^{\phi_1+\phi_F/2}\rho_t\,\mathrm{d}\phi_t\,\mathrm{e}^{i\xi_m[\rho_1\rho_t/2(\rho_1-\rho_t)](\phi_t-\phi_1)^2}\int_{\phi_2-\phi_F/2}^{\phi_2+\phi_F/2}\rho_{t'}\,\mathrm{d}\phi_{t'}\,\mathrm{e}^{-i\xi_{m'}[\rho_2\rho_{t'}/2(\rho_2-\rho_{t'})](\phi_{t'}-\phi_2)^2} \\
&\times\frac{4\pi^2}{k(z_t)k(z_{t'})d(z_t)d(z_{t'})}\frac{1}{d^2(z_0)}\frac{1}{\sqrt{\xi_m\xi_{m'}^*}}\frac{1}{\sqrt{\xi_n\xi_{n'}^*}}\frac{1}{\sqrt{(\rho_1-\rho_t)(\rho_2-\rho_t')\rho_t\rho_{t'}}} \\
&\times u_m(z_1)u_{m'}^*(z_2)u_n(z_0)u_{n'}^*(z_0) \\
&\times[N_m^{(1)}(z_t)N_n^{(1)}(z_t)\mathrm{e}^{i\Re(\gamma_m+\gamma_n)z_t}s_{\mathbf{r_t}}(\pi-\alpha_m,\beta_s(\phi_1,\phi_t);\alpha_n,\phi_t) \\
&-N_m^{(2)}(z_t)N_n^{(1)}(z_t)\mathrm{e}^{i\Re(-\gamma_m+\gamma_n)z_t}s_{\mathbf{r_t}}(\alpha_m,\beta_s(\phi_1,\phi_t);\alpha_n,\phi_t) \\
&-N_m^{(1)}(z_t)N_n^{(2)}(z_t)\mathrm{e}^{i\Re(\gamma_m-\gamma_n)z_t}s_{\mathbf{r_t}}(\pi-\alpha_m,\beta_s(\phi_1,\phi_t);\pi-\alpha_n,\phi_t) \\
&+N_m^{(2)}(z_t)N_n^{(2)}(z_t)\mathrm{e}^{i\Re(-\gamma_m-\gamma_n)z_t}s_{\mathbf{r_t}}(\alpha_m,\beta_s(\phi_1,\phi_t);\pi-\alpha_n,\phi_t)] \\
&\times[N_{m'}^{*(1)}(z_{t'})N_{n'}^{*(1)}(z_{t'})\mathrm{e}^{i\Re(-\gamma_{m'}-\gamma_{n'})z_{t'}}s_{\mathbf{r_{t'}}}(\pi-\alpha_{m'},\beta_s(\phi_2,\phi_{t'});\alpha_{n'},\phi_{t'})
\end{aligned}$$

$$- N_{m'}^{*(2)}(z_{t'})N_{n'}^{*(1)}(z_{t'})\mathrm{e}^{i\Re(\gamma_{m'}-\gamma_{n'})z_{t'}}s_{\mathbf{r}_{t'}}(\alpha_{m'},\beta_s(\phi_2,\phi_{t'});\alpha_{n'},\phi_{t'})$$
$$- N_{m'}^{*(1)}(z_{t'})N_{n'}^{*(2)}(z_{t'})\mathrm{e}^{i\Re(-\gamma_{m'}+\gamma_{n'})z_{t'}}s_{\mathbf{r}_{t'}}(\pi-\alpha_{m'},\beta_s(\phi_2,\phi_{t'});\pi-\alpha_{n'},\phi_{t'})$$
$$+ N_{m'}^{*(2)}(z_{t'})N_{n'}^{*(2)}(z_{t'})\mathrm{e}^{i\Re(\gamma_{m'}+\gamma_{n'})z_{t'}}s_{\mathbf{r}_{t'}}(\alpha_{m'},\beta_s(\phi_2,\phi_{t'});\pi-\alpha_{n'},\phi_{t'})]$$
$$\times\ \mathrm{e}^{i\Re\{\xi_m(\rho_1-\rho_t)-\xi_{m'}(\rho_2-\rho_{t'})\}}\mathrm{e}^{i\Re\{\xi_n\rho_t-\xi_{n'}\rho_{t'}\}}$$
$$\times\ \mathrm{e}^{-\Im\{\xi_m(\rho_1-\rho_t)+\xi_{m'}(\rho_2-\rho_{t'})\}}\mathrm{e}^{-\Im\{\xi_n+\xi_{n'}\}\rho_t}\mathrm{e}^{-\Im\{(\gamma_m+\gamma_n)z_t+(\gamma_{m'}+\gamma_{n'})z_{t'}\}}\Bigg\rangle. \tag{106}$$

Equation (106) cannot be further evaluated unless the cross-correlation of the scatter function densities at $\mathbf{r_t}$ and $\mathbf{r_{t'}}$ is known. As before, we examine the two cases. The first is when inhomogeneities are fully correlated and the second is when they are uncorrelated within the Fresnel width for both receivers.

1. Scatterers fully correlated within Fresnel width

This analysis is applicable to shells where $l_y > Y_{\mathrm{F}}(\rho_p,\rho_s)$ or $|\rho_s-\rho_p/2| > (\rho_p/2)\sqrt{1-4l_y^2/\lambda\rho}$ for $p=1,2$. The exponential coefficient of field covariance for the two receiver azimuths is found to be

$$\mu_{n,1,2}^{\mathrm{cor}}(\rho_s)=\sum_m \frac{1}{|\xi_m|}\int_0^\infty \mathrm{d}z_t\int_0^\infty \mathrm{d}z_{t'}\frac{l_x(\rho_s,z_t,z_{t'})}{\xi_m}\frac{4\pi^2}{k(z_t)k(z_{t'})d(z_t)d(z_{t'})}C_{s,s}(\rho_s,z_t,z_{t'},m,n), \tag{107}$$

by setting $\rho_1=\rho_2=\rho$ in Eq. (106) and following an approach similar to that in Sec. IV B 1. Here $C_{s,s}(\rho_s,z_t,z_{t'},m,n)$ is given by Eq. (72), but with the covariance of the scatter function density defined as

$$\mathrm{Cov}(s_{\rho_s,z_t}(m_u;n_d),s_{\rho_s,z_{t'}}(m_u;n_d))$$
$$=\langle s_{\rho_s,z_t}(\pi-\alpha_m,\phi_1;\alpha_n,\phi_1)s^*_{\rho_s,z_{t'}}(\pi-\alpha_m,\phi_2;\alpha_n,\phi_2)\rangle-\langle s_{\rho_s,z_t}(\pi-\alpha_m,\phi_1;\alpha_n,\phi_1)\rangle\langle s^*_{\rho_s,z_{t'}}(\pi-\alpha_m,\phi_2;\alpha_n,\phi_2)\rangle. \tag{108}$$

The solutions for receivers at different ranges and azimuths are then obtained by substituting Eq. (107) into Eqs. (104) and (105).

2. Uncorrelated scatterers within the Fresnel width

This analysis is applicable to shells where $l_y < Y_F(\rho_p,\rho_s)$ or $|\rho_s-\rho_p/2| < (\rho_p/2)\sqrt{1-4l_y^2/\lambda\rho_p}$ for $p=1,2$. The scatter function densities centered at $\mathbf{r_t}$ as well as $\mathbf{r_{t'}}$ are assumed to be fully correlated if they fall within both a coherence area A_c of each other and the overlapping active regions of the given shell for the two receivers. Let $W_{\mathrm{overlap}}(\phi_t)$ be an azimuthal window function that describes this overlap region,

$$W_{\mathrm{overlap}}(\phi_t)=\begin{cases}1, & \text{if } (\phi_1-\phi_F/2<\phi_t<\phi_1+\phi_F/2)\,\&\,(\phi_2-\phi_F/2<\phi_t<\phi_2+\phi_F/2),\\ 0, & \text{otherwise}.\end{cases} \tag{109}$$

The coefficient of modal field variance for the two receiver azimuths becomes

$$\mu_n^{\mathrm{uncor}}(\rho_s)=\sum_m\sqrt{\frac{\rho}{2\pi\xi_m\rho_s(\rho-\rho_s)}}\frac{1}{|\xi_m|}\mathcal{B}(\rho,\rho_s,\phi_1,\phi_2)\int_0^\infty \mathrm{d}z_t\int_0^\infty \mathrm{d}z_{t'}\,A_c(\rho_s,z_t,z_{t'})\frac{4\pi^2}{k(z_t)k(z_{t'})d(z_t)d(z_{t'})}C_{s,s}(\rho_s,z_t,z_{t'},m,n), \tag{110}$$

by setting $\rho_1=\rho_2=\rho$ in Eq. (106) and following an analysis similar to that in Sec. IV B 2. Here,

$$\mathcal{B}(\rho,\rho_s,\phi_1,\phi_2)=\mathrm{e}^{i\xi_m[\rho\rho_s/2(\rho-\rho_s)](\phi_1^2-\phi_2^2)}\sqrt{\frac{\xi_m\rho}{2\pi\rho_s(\rho-\rho_s)}}\int_0^{2\pi}W_{\mathrm{overlap}}(\phi_t)\mathrm{e}^{i\xi_m\rho_s\phi_t[\rho/2(\rho-\rho_s)](\phi_2-\phi_1)}\rho_s\,\mathrm{d}\phi_t, \tag{111}$$

and $C_{z_t,z_{t'}}(n,m)$ is given by Eq. (72), but with the covariance of the scatter function defined by

$$\mathrm{Cov}(s_{z_t}(m_\mathrm{u};n_\mathrm{d}),s_{z_{t'}}(m_\mathrm{u};n_\mathrm{d})) = \left\langle s_{z_t}\left(\pi-\alpha_m,\beta_s\left(\phi_1,\frac{\phi_1+\phi_2}{2}\right);\alpha_n,\frac{\phi_1+\phi_2}{2}\right)s^*_{z_{t'}}\left(\pi-\alpha_m,\beta_s\left(\phi_2,\frac{\phi_1+\phi_2}{2}\right);\alpha_n,\frac{\phi_1+\phi_2}{2}\right)\right\rangle - \left\langle s_{z_t}\left(\pi-\alpha_m,\beta_s\left(\phi_1,\frac{\phi_1+\phi_2}{2}\right);\alpha_n,\frac{\phi_1+\phi_2}{2}\right)\right\rangle\left\langle s^*_{z_{t'}}\left(\pi-\alpha_m,\beta_s\left(\phi_2,\frac{\phi_1+\phi_2}{2}\right);\alpha_n,\frac{\phi_1+\phi_2}{2}\right)\right\rangle. \quad (112)$$

The solutions for receivers at different ranges and azimuths are then obtained by substituting Eq. (110) into Eqs. (104) and (105). For two receivers that are colocated so that $\phi_1=\phi_2=\phi$, $\mathcal{B}(\rho,\rho_s,\phi,\phi)=1$ and Eq. (110) reduces to Eq. (77) for a single receiver.

VII. CONCLUSION

Compact analytic expressions are derived for the mean, mutual intensity, and spatial covariance of the acoustic field foward propagated through a stratified ocean waveguide containing 3-D random surface or volume inhomogeneities. They include the accumulated effect of multiple forward scattering through the random inhomogeneous waveguide. They are given as a modal solution in terms of the parameters necessary to describe the incident field as well as the mean and spatial covariance of the medium's inhomogeneities. They are applicable to a broad range of remote sensing and communication problems in the ocean. This includes propagation through bubble clouds, fish schools, internal waves, turbulence, and seabed anomalies in waveguides with rough boundaries.

Multiple scattering through the randomly inhomogeneous medium leads to a mean field where each mode propagates with a new horizontal wave number that depends on the expected scatter function density of the random medium. The new wave number describes attenuation and dispersion induced by the medium's randomness, including potential mode coupling along the propagation path. Expressions for the mutual intensity and spatial covariance of the forward field depend on both the random medium's expected modal extinction density as well as the covariance of its scatter function density, which couples each mode to all other modes due to multiple forward scattering. These are used to analytically show that 3-D scattering effects can become important for scatterers at ranges where the Fresnel width exceeds the medium's local cross-range coherence length. The expressions can also be applied to determine how the coherence of an acoustic signal received by an array of arbitrary configuration is degraded by random multiple forward scattering through the fluctuating ocean.

APPENDIX A: SCATTER FUNCTION DENSITY

Discretion must be used in choosing a parametrization for the scatter function density that properly describes the random characteristics of the medium and their relationship to the scattering process, which must conserve energy. Here we present representations of the scatter function density $s_{\mathbf{r_t}}(\alpha,\beta;\alpha_i,\beta_i)$ for both continuous and discrete random scatterers.

1. Continuous random inhomogeneities

Random inhomogeneitites in the acoustic medium are in essence stochastic spatial variations in density and compressibility. Continuous random inhomogeneities in ocean waveguides include turbulence, internal waves, sea-surface and seabed roughness, as well as anomalies in the seabed.

a. Local 3-D stationary random process, potentially nonisotropic

If the inhomogeneities obey a 3-D stationary random process within the single scatter shell, the coherence volume can be defined as

$$V_c(\rho_s,z_t) = \frac{\int_0^{2\pi}\int_0^{\pi}\int_0^{R_{\max}} |C_{\mathcal{FF}}(R,\Omega_1,\Omega_2,\rho_s,z_t)|^2 R^2 \sin\Omega_1\, \mathrm{d}R\, \mathrm{d}\Omega_1\, \mathrm{d}\Omega_2}{|C_{\mathcal{FF}}(0,0,0,\rho_s,z_t)|^2}, \quad (A1)$$

where the scattering density $\mathcal{F}$ of an elemental volume of the randomly inhomogeneous medium can be defined by the Rayleigh-Born single scatter approximation as

$$\Re\{\mathcal{F}(\alpha,\beta,\alpha_i,\beta_i)\} = \frac{k^3}{4\pi}(\Gamma_\kappa(\mathbf{r_t}) + \eta(\mathbf{k},\mathbf{k}_i)\Gamma_d(\mathbf{r_t})), \quad (A2)$$

where Γ_κ is the fractional change in compressibility and Γ_d is the fractional change in density of the local inhomogeneity with respect to the original homogeneous medium,[4] and

$$\eta(\mathbf{k},\mathbf{k}_i) = \frac{\mathbf{k}_i\cdot\mathbf{k}}{k^2} \quad (A3)$$

is the cosine of the angle between the incident and scattered plane wave directions. When density changes are important to acoustic scattering, perturbation expansions in *sound speed* and density alone are typically not sufficient to describe even first-order effects. To include all necessary effects, expansions in *compressibility* and density must be made.[3]

Here the covariance of the elemental scattering density $\mathrm{Cov}(\mathcal{F}_{\mathbf{r_t}},\mathcal{F}^*_{\mathbf{r_{t'}}})=C_{\mathcal{FF}}(R,\Omega_1,\Omega_2,\rho_s,z_t)$ depends only on the separation $\mathbf{R}=(R,\Omega_1,\Omega_2)=\mathbf{r_t}-\mathbf{r'_t}$, where $R=\sqrt{(x_t-x_{t'})^2+(y_t-y_{t'})^2+(z_t-z_{t'})^2}$. The coherence radius for any direction specified by the elevation angle Ω_1 and azimuth angle Ω_2 is given by

$$l_c^3(\Omega_1,\Omega_2,\rho_s,z_t)=\frac{3\int_0^{R_{\max}}|C_{\mathcal{FF}}(R,\Omega_1,\Omega_2,\rho_s,z_t)|^2\,\mathrm{d}R}{|C_{\mathcal{FF}}(0,0,0,\rho_s,z_t)|^2}. \tag{A4}$$

The coherence lengths for the random process in the x, y, and z directions are then given by $l_x(\rho_s,z_t)=l_c(\Omega_1=\pi/2,\Omega_2=0,\rho_s,z_t)+l_c(\Omega_1=\pi/2,\Omega_2=\pi,\rho_s,z_t)$, $l_y(\rho_s,z_t)=l_c(\Omega_1=\pi/2,\Omega_2=\pi/2,\rho_s,z_t)+l_c(\Omega_1=\pi/2,\Omega_2=3\pi/2,\rho_s,z_t)$, and $l_z(\rho_s,z_t)=l_c(\Omega_1=0,\Omega_2=0,\rho_s,z_t)+l_c(\Omega_1=\pi,\Omega_2=0,\rho_s,z_t)$. In Eq. (A1), $R_{\max}$ should be larger than l_c.

The 3-D delta-function covariance assumed in Eq. (93) is consistent with the interpretion that

$$s_{\rho_s,z_t}(\alpha,\beta,\alpha_i,\beta_i)=\frac{1}{V_c}\iiint_{V_c}\mathcal{F}_{\mathbf{r_t+u}}(\alpha,\beta,\alpha_i,\beta_i)\mathrm{e}^{i(\mathbf{k_i}-\mathbf{k_s})\cdot\mathbf{u}}\mathrm{d}^3\mathbf{u}, \tag{A5}$$

be used on the right-hand side of Eq. (93), given continuous inhomogeneities. Here $\mathbf{k_i}$ and $\mathbf{k_s}$ are the incident and scattered wave number vectors, and $\mathbf{u}=(u_x,u_y,u_z)$ are locations within the coherence volume relative to $\mathbf{r_t}$. The corresponding free-space plane wave scatter function S_{z_t} for the coherence volume is then obtained by multiplying Eq. (A5) by V_c.

b. Local 2-D stationary random process, potentially nonisotropic

Given the 2-D delta-function covariance assumed in Eq. (75), the interpretation,

$$s_{\rho_s,z_t}(\alpha,\beta,\alpha_i,\beta_i)=\frac{1}{A_c}\iint_{A_c}\mathcal{F}_{\mathbf{r_t+u}}(\alpha,\beta,\alpha_i,\beta_i)\mathrm{e}^{i(\boldsymbol{\xi}_\mathbf{i}-\boldsymbol{\xi}_\mathbf{s})\cdot\mathbf{u}}\mathrm{d}^2\mathbf{u} \tag{A6}$$

can be made on the right-hand side of Eq. (75) for continuous inhomogeneities. Here $\boldsymbol{\xi}_\mathbf{i}$ and $\boldsymbol{\xi}_\mathbf{s}$ are the incident and scattered horizontal wave number vectors and $\mathbf{u}=(u_x,u_y)$ are horizontal locations within the coherence area relative to $\boldsymbol{\rho}_t$.

Here the horizontal coherence area A_c is defined under the assumption that the inhomogeneities obey a stationary random process in the horizontal within the Fresnel width in each single scatter shell. The horizontal covariance of their scattering densities can then be expressed as $\mathrm{Cov}(\mathcal{F}_{\mathbf{r_t}},\mathcal{F}^*_{\mathbf{r_{t'}}})=C_{\mathcal{FF}}(P,\Omega,\rho_s,z_t,z_{t'})$, which depends on the separation between their horizontal coordinates, $P=\sqrt{(x_t-x_{t'})^2+(y_t-y_{t'})^2}$, and $\Omega=\tan^{-1}[(y_t-y_{t'})/(x_t-x_{t'})]$ but not on their absolute position within the shell.

The generalized parametric coherence area for the random process in the horizontal is then defined as[55]

$$A_c(\rho_s,z_t,z_{t'})=\frac{\int_0^{2\pi}\int_0^{P_{\max}}|C_{\mathcal{FF}}(P,\Omega,\rho_s,z_t,z_{t'})|^2P\,\mathrm{d}P\,\mathrm{d}\Omega}{|C_{\mathcal{FF}}(0,0,\rho_s,z_t,z_{t'})|^2}, \tag{A7}$$

with coherence radius $l_c(\Omega,\rho_s,z_t,z_{t'})$ for any direction Ω, given by

$$l_c^2(\Omega,\rho_s,z_t,z_{t'})=\frac{2\int_0^{P_{\max}}|C_{\mathcal{FF}}(P,\Omega,\rho_s,z_t,z_{t'})|^2P\,\mathrm{d}P}{|C_{\mathcal{FF}}(0,0,\rho_s,z_t,z_{t'})|^2}. \tag{A8}$$

The coherence lengths for the random process in the x and y directions are then given by $l_x(\rho_s,z_t,z_{t'})=l_c(\Omega=0,\rho_s,z_t,z_{t'})+l_c(\Omega=\pi,\rho_s,z_t,z_{t'})$ and $l_y(\rho_s,z_t,z_{t'})=l_c(\Omega=\pi/2,\rho_s,z_t,z_{t'})+l_c(\Omega=3\pi/2,\rho_s,z_t,z_{t'})$, where $P_{\max}$ in Eq. (A7) is larger than l_c.

2. Discrete inhomogeneities or particles

Consider a volume of space V centered at location $\mathbf{r}_t$ containing discrete inhomogeneities or particles. Each particle may be large compared to the waveglength and have arbitrary shape and material properties. Examples of such discrete scatterers include fish, marine mammals, bubbles, and underwater vehicles. The location of the qth particle in this volume is $\mathbf{r}_{t,q}=\mathbf{r}_t+\mathbf{u}_q$, where $\mathbf{u}_q$ is its displacement from the volume center $\mathbf{r}_t$. The scatter function of the qth particle relative to the volume center is $S_q(\mathbf{r}_t)\mathrm{e}^{i(\mathbf{k}_i-\mathbf{k}_s)\cdot\mathbf{u}_q}$, where $S_q(\mathbf{r}_t)$ is its scatter function, $\mathbf{k}_i$ and $\mathbf{k}_s$ are the incident and scattered plane wave vectors, respectively. For discussions in this section, we suppress the angular dependence of the scatter function to abbreviate the notation.

When the single scatter approximation is valid, the expected total scatter function $\langle S_T(\mathbf{r}_t)\rangle$ of the volume V is then

$$\langle S_T(\mathbf{r}_t)\rangle=\int\int\int\sum_q^N S_q(\mathbf{r}_t)\mathrm{e}^{i(\mathbf{k}_i-\mathbf{k}_s)\cdot\mathbf{u}_q}p(\mathbf{u}_q,S_q|N)\times p(N)\mathrm{d}^3\mathbf{u}_q\,\mathrm{d}S_q\,\mathrm{d}N, \tag{A9}$$

where $p(\mathbf{u}_q,S_q|N)$ is the probability density of finding the qth particle at location $\mathbf{u}_q$ with scatter function amplitude S_q given that there are N particles in the volume, and $p(N)$ is the probability density of finding N particles in this volume.

If the particles in V are each identically distributed in space and their spatial distribution is uniform and independent of the scatter function amplitude and the number of particles in the volume, then $p(\mathbf{u}_q,S_q|N)=p(\mathbf{u}_q)p(S_q|N)$, and $p(\mathbf{u}_q)=1/V$. Equation (A9) then becomes

$$\langle S_T(\mathbf{r}_t)\rangle = U\int\int\sum_q^N S_q(\mathbf{r}_t)p(S_q|N)p(N)\mathrm{d}S_q\,\mathrm{d}N, \qquad \text{(A10)}$$

where

$$U=\int \frac{1}{V}\mathrm{e}^{i(\mathbf{k}_i-\mathbf{k}_s)\cdot\mathbf{u}_q}\mathrm{d}^3\mathbf{u}_q \qquad \text{(A11)}$$

is an elementary integral over V that is a function of $\mathbf{k}_i$ and $\mathbf{k}_s$. For forward scatter in free space, $U=1$, since $\mathbf{k}_i$ and $\mathbf{k}_s$ are identical in the forward direction. This point was essentially made long ago by Rayleigh.[10] In a multimodal waveguide, however, U will not necessarily be unity, even for scattering in the forward azimuth due to variations in modal elevation angles. It is always approximately unity when the dimensions of V can be made small compared to the acoustic wavelength, assuming due attention is paid to the actual size of the particles.

If the particles are also identically distributed in their scatter function amplitude, and N is large, the total scatter function of the volume from Eq. (A10) then reduces to

$$\langle S_T(\mathbf{r}_t)\rangle = U\int N\overline{S(\mathbf{r}_t|N)}p(N)\mathrm{d}N, \qquad \text{(A12)}$$

$$=U\langle N\overline{S(\mathbf{r}_t|N)}\rangle, \qquad \text{(A13)}$$

where $\overline{S(\mathbf{r}_t|N)}=\int S_q(\mathbf{r}_t)p(S_q|N)\mathrm{d}S_q$ is the expected scatter function amplitude of each of the identically distributed particles, given that there are a total of N particles in the volume.

The expected scatter function density at location $\mathbf{r}_t$ is then given by

$$\langle s_{\mathbf{r_t}}\rangle = \frac{\langle S_T(\mathbf{r}_t)\rangle}{V} = U\langle n_V\overline{S(\mathbf{r}_t|n_V)}\rangle, \qquad \text{(A14)}$$

where $n_V=N/V$ is the number of particles per unit volume.

The second moment of the scatter function of the particles within V is

$$\langle S_T(\mathbf{r}_t)S_T^*(\mathbf{r}_t)\rangle = \int\int\int\int\int\sum_q^N\sum_l^N S_q(\mathbf{r}_t)S_l^*(\mathbf{r}_t)\mathrm{e}^{i(\mathbf{k}_i-\mathbf{k}_s)\cdot\mathbf{u}_q} \times \mathrm{e}^{-i(\mathbf{k}_i-\mathbf{k}_s)\cdot\mathbf{u}_l}p(\mathbf{u}_q,\mathbf{u}_l,S_q,S_l|N)p(N) \times \mathrm{d}\mathbf{u}_q\,\mathrm{d}\mathbf{u}_l\,\mathrm{d}S_q\,\mathrm{d}S_l\,\mathrm{d}N. \qquad \text{(A15)}$$

If we assume that the scatter functions and positions of the particles are uncorrelated, Eq. (A15) then reduces to

$$\langle S_T(\mathbf{r}_t)S_T^*(\mathbf{r}_t)\rangle = \int\sum_q^N\sum_l^N\left([\mathrm{Var}(S_q(\mathbf{r}_t|N))+(1-|U|^2)|\overline{S_q(\mathbf{r}_t|N)}|^2]\delta_{ql} + |U|^2\overline{S_q(\mathbf{r}_t|N)}\ \overline{S_l^*(\mathbf{r}_t|N)}\right)p(N)\mathrm{d}N, \qquad \text{(A16)}$$

where $\mathrm{Var}(S_q(\mathbf{r}_t|N))=\int|S_q(\mathbf{r}_t)|^2p(S_q|N)\mathrm{d}S_q-|\overline{S_q(\mathbf{r}_t|N)}|^2$, and the second term in the square bracket is a variance component that arises solely from randomness in the particle position.

Since the scatterers are also taken to be identically distributed, and N is large, Eq. (A16) further simplifies to

$$\langle S_T(\mathbf{r}_t)S_T^*(\mathbf{r}_t)\rangle = \int (N\,\mathrm{Var}(S(\mathbf{r}_t|N))+(1-|U|^2)N|\overline{S(\mathbf{r}_t|N)}|^2 + |U|^2N^2|\overline{S(\mathbf{r}_t|N)}|^2)p(N)\mathrm{d}N, \qquad \text{(A17)}$$

$$=\langle N\,\mathrm{Var}(S(\mathbf{r}_t|N))\rangle+(1-|U|^2)\langle N|\overline{S(\mathbf{r}_t|N)}|^2\rangle + |U|^2\langle N^2|\overline{S(\mathbf{r}_t|N)}|^2\rangle. \qquad \text{(A18)}$$

The second moment of the scatter function density then becomes

$$\langle|s_{\mathbf{r_t}}|^2\rangle = \frac{\langle|S_T(\mathbf{r}_t)|^2\rangle}{(V)^2} = \frac{1}{V}\left[\langle n_V\,\mathrm{Var}(S(\mathbf{r}_t|N))\rangle+(1-|U|^2)\langle n_V|\overline{S(\mathbf{r}_t|n_V)}|^2\rangle\right] + |U|^2\langle n_V^2|\overline{S(\mathbf{r}_t|n_V)}|^2\rangle, \qquad \text{(A19)}$$

and its variance is

$$\mathrm{Var}(s_{\mathbf{r_t}})=\langle|s_{\mathbf{r_t}}|^2\rangle-|\langle s_{\mathbf{r_t}}\rangle|^2, \qquad \text{(A20)}$$

$$=\frac{1}{V}[\langle n_V\,\mathrm{Var}(S(\mathbf{r}_t|N))\rangle+(1-|U|^2)\langle n_V|\overline{S(\mathbf{r}_t|n_V)}|^2\rangle] + |U|^2\langle n_V^2|\overline{S(\mathbf{r}_t|n_V)}|^2\rangle-|U|^2|\langle n_V\overline{S(\mathbf{r}_t|n_V)}\rangle|^2. \qquad \text{(A21)}$$

If scattering by the particles is independent of the number of particles per unit volume in the medium, the mean and variance of the scatter function density then, respectively, become

$$\langle s_{\mathbf{r_t}}\rangle = U\langle n_V\rangle\langle S(\mathbf{r}_t)\rangle, \qquad \text{(A22)}$$

and

$$\mathrm{Var}(s_{\mathbf{r_t}})=\frac{1}{V}[\langle n_V\rangle\mathrm{Var}(S(\mathbf{r}_t))+(1-|U|^2)\langle n_V\rangle|\langle S(\mathbf{r}_t)\rangle|^2] + |U|^2\,\mathrm{Var}(n_V)|\langle S(\mathbf{r}_t)\rangle|^2. \qquad \text{(A23)}$$

For forward scatter in free space $|U|^2=1$ and the variance of the scatter function density in Eq. (A23) reduces to

$$\mathrm{Var}(s_{\mathbf{r_t}})=\frac{1}{V}\langle n_V\rangle\mathrm{Var}(S(\mathbf{r}_t))+\mathrm{Var}(n_V)|\langle S(\mathbf{r}_t)\rangle|^2. \qquad \text{(A24)}$$

Under the assumptions and approximations leading to Eq. (A24), the variance of the scatter function density for discrete scatterers in the forward direction in free space is zero when both the variance of the scatter function of an individual particle and the variance of the number density of particles are zero. Otherwise, the scatter function variance and, consequently, the field variance cannot be zero. This result is especially intuitive in forward scatter, where changes in the configuration of a fixed number of like and uncorrelated particles within a farfield volume can have no effect on the forward field. We again note that essentially this same point was made long ago by Rayleigh[10] and can be obtained in the appropriate limiting case of Eqs. (7.3.13) and (7.3.18) in the Tsang, Kong, and Ding[56] analysis for omnidirectional scatterers.

It is interesting that in Twersky's formulation of forward scattering through a slab of discrete scatterers, he implicitly assumed the variances of the scatter function and the number density to be zero and also obtains zero variance for the forward field for his q small, which corresponds to what we refer to here as the direct wave, as can be seen in Sec. 14-6 of Ishimaru.[18] It is not clear that Twersky's formulation is valid when his q is not small, given the numerous small angle approximations he makes for the scattered field about the incident direction.

Similarly, the cross-correlation of the scatter function densities centered at $\mathbf{r}_{t1}$ and $\mathbf{r}_{t2}$ can be approximated as

$$\langle s_{\mathbf{r_{t1}}} s^*_{\mathbf{r_{t2}}} \rangle = \frac{\langle S_T(\mathbf{r}_{t1}) S_T^*(\mathbf{r}_{t2})\rangle}{(V_c)^2} = V_c \delta(\mathbf{r}_{t1}-\mathbf{r}_{t2}) \left\{ \frac{\langle n_V(\mathbf{r}_{t1})\rangle}{V_c} \mathrm{Var}(S(\mathbf{r}_{t1})) + \left[\frac{\langle n_V(\mathbf{r}_{t1})\rangle}{V_c} (1-|U(\mathbf{r}_{t1})|^2) + |U(\mathbf{r}_{t1})|^2 \,\mathrm{Var}(n_V(\mathbf{r}_{t1}))\right] |\langle S(\mathbf{r}_{t1})\rangle|^2 \right\} + U(\mathbf{r}_{t1}) U^*(\mathbf{r}_{t2}) \langle n_V(\mathbf{r}_{t1})\rangle \langle n_V(\mathbf{r}_{t2})\rangle \langle S(\mathbf{r}_{t1})\rangle \langle S^*(\mathbf{r}_{t2})\rangle, \tag{A25}$$

if the coherence volume V_c falls within the active region of a given single scatter shell. Here the coherence volume is defined entirely by the spatial covariance of the number density through

$$V_c = \frac{\iiint |C_{n_V n_V}(\mathbf{R})|^2 \, d^3\mathbf{R}}{|C_{n_V n_V}(0)|^2}, \tag{A26}$$

where the covariance of the number density $\mathrm{Cov}(n_{V\mathbf{r_t}}, n_{V\mathbf{r_{t'}}}) = C_{n_V n_V}(\mathbf{R})$ depends only the separation $\mathbf{R} = \mathbf{r_t} - \mathbf{r'_t}$.

If $|U|^2 V_c \,\mathrm{Var}(n_V)$ is negligible, then

$$\langle s_{\mathbf{r_{t1}}} s^*_{\mathbf{r_{t2}}} \rangle = \frac{\langle S_T(\mathbf{r}_{t1}) S_T^*(\mathbf{r}_{t2})\rangle}{(V)^2} = \langle n_V(\mathbf{r}_{t1})\rangle \delta(\mathbf{r}_{t1}-\mathbf{r}_{t2}) \{ \mathrm{Var}(S(\mathbf{r}_{t1})) + [(1-|U(\mathbf{r}_{t1})|^2)] |\langle S(\mathbf{r}_{t1})\rangle|^2 \} + U(\mathbf{r}_{t1}) U^*(\mathbf{r}_{t2}) \langle n_V(\mathbf{r}_{t1})\rangle \langle n_V(\mathbf{r}_{t2})\rangle \langle S(\mathbf{r}_{t1})\rangle \langle S^*(\mathbf{r}_{t2})\rangle, \tag{A27}$$

as long as V can be limited to the active region and have the corresponding number of particles N within it be large. Equations (A25) or (A27) can be used in Eq. (93) for scattering from discrete particles or objects that decorrelate in three dimensions. The spatial correlation of scattering from objects that decorrelate in only one or two dimensions can be readily obtained by analogous means.

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