The effect of saturated transmission scintillation on ocean acoustic intensity measurements

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Coherence theory is used to analyze the statistical properties of ocean-acoustic intensity fluctuations measured after saturated multipath propagation. Previous analyses in this area have been implicitly limited to certain special cases for which the time-bandwidth product of the field received from a given source is unity. In this paper, the statistical description is extended and generalized to be a function of measurement time and temporal coherence. As a result, the well known 5.6-dB transmission loss (TL) standard deviation of Dyer is found to be a special case of a more general TL standard deviation that approximates 4.34 sqrt$(1/\mu)$ dB when the time-bandwidth product μ is large. Therefore, the TL standard deviation approaches zero for increasing μ, as it must in the deterministic limit of an arbitrarily large sample size. A similar generalization is obtained for the TL mean, from which it is found that the sonar equation must be corrected for a μ-dependent bias that vanishes in the deterministic limit of large μ. Additionally, asymptotic analysis shows that intensity statistics in the saturated region converge to a log-normal distribution, where $\mu>4$ is typically sufficient for the log-normal approximation to be made.

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INTRODUCTION

As long ago as World War II, marine physicists under the National Defense Research Committee[1] observed that natural disturbances, such as underwater turbulence and passing surface or internal gravity waves, often place the ocean-acoustic waveguide in such a state of flux that a signal, deterministic when transmitted from the source, becomes fully randomized after propagating only several channel depths away in range to a receiver. When not properly accounted for, such randomization can severely degrade the accuracy of an experimental measure, as well as any parameter estimates based upon this measure. However, with the knowledge that the field is fully randomized, and thereby can be described by a circular complex Gaussian random (CCGR) variable,[2] many useful statistical properties of subsequent intensity measurements and parameter estimates can be readily deduced by respective applications of coherence theory and estimation theory.

The CCGR field assumption has a long history. In the analysis of random signals and noise, it has been legitimately made when the central limit theorem applies, such as in the scattering of radiation from fluctuating targets[3] and surfaces with wavelength-scale roughness.[4] In the present context, it is the basis for the Rayleigh scatter channel that is not only frequently cited in communication theory,[5] but has also been used to describe the saturated region[6] of multipath propagation in the ocean for many years.[1] Previous analyses in this area, however, have been implicitly limited to certain special cases for which the time-bandwidth product of the field received from a given source is unity. In this paper, the statistical formulation is extended and generalized to be a function of measurement time and temporal coherence. This more general framework is therefore highly relevant to modern ocean-acoustic sonar and communication systems that employ time-bandwidth products exceeding unity, or average over many independent samples. Such averaging is often essential when the goal is to reduce the variance of a measurement, or subsequent parameter estimate, enough for the error to fall within a tolerable threshold.

Much of the literature in ocean-acoustic transmission scintillation[6–11] closely follows or is based upon the classic analysis of Dyer,[12] who, like his predecessor Bergmann,[1] concluded that in many circumstances a Rayleigh scatter channel can be used to describe the random fluctuations of measured intensity. In order to extend this analysis to the case of variable time-bandwidth product, some specific results from Dyer's ''Statistics of sound propagation in the ocean''[12] are used as a guide line. First, techniques well established in coherence theory are used to formulate the statistical properties of averaged intensity in terms of measurement time and temporal coherence. This is done in Sec. I where it is also shown that the ''short-time average'' of Dyer corresponds to a measurement that has a time-bandwidth product of unity and is therefore by definition instantaneous. In Sec. II, the distribution for the logarithm of averaged intensity is given as a function of measurement time and temporal coherence. As a result, Dyer's well known 5.6 dB transmission loss (TL) standard deviation and 2.5-dB augmentative bias in mean TL are found to be valid only for the special case of an instantaneous measurement. The general TL standard deviation and mean have forms that, respectively, approach zero and $-10\log$(the mean-square transmission), as they must, in the deterministic limit of increasing time-bandwidth product. Further analysis along this line indicates that the logarithm of averaged intensity has a distribution that rapidly converges to a Gaussian for time-bandwidth products exceeding four. The significance of this particular finding is that, by invocation of the central limit theorem for sufficiently large time-bandwidth products, intensity statistics in the saturated region can legitimately be described as log-normal, which runs counter to some previous suggestions.[12]

Additionally, certain special assumptions were made in the derivation of a well-known probability distribution for the ''noise of multiple distant sources'' [12] that limits its general usefulness. Specifically, the *intensity* contribution of each source was assumed to be independent, exponentially distributed, and comprised of a single tone spectrally disjoint from the simultaneously measured tones of the other sources. These assumptions imply that the time-bandwidth product of the total received field must equal the number of independent sources, and the measurement period must be such that the spectral contribution of each tone is resolvable according to the Rayleigh criterion.[13] In Sec. III of the present paper, the more general assumption is made that the measured *fields* from the independent sources are independent CCGR variables, and no restriction is made on their spectra except that, for practical considerations, they have finite bandwidth. The intensity distribution for the ''noise of multiple distant sources'' derived under this more general assumption is given as a function of the measurement time and temporal coherence of the total received field. Similar differences between the distributions for a ''signal plus noise'' [12] derived in previous work and those derived here are discussed.

The statistics of averages of independent intensity samples are also investigated. Such averages are widely used in a variety of ocean acoustic applications. For example, in displaying the beamformed output of a hydrophone array, it is common practice to reduce the variance by averaging the uncorrelated intensities received on adjacent nonoverlapping beams.[14] Similarly, it is sometimes convenient to average independent multipath arrivals that are temporally disjoint, or to average independent measurements of backscatter to reduce the variance in scattering strength or target strength estimation.[14–15] In Sec. IV, the probability distributions for such amalgamated intensity measurements are investigated. The probability distribution for the difference between two intensity measurements is then given in Sec. V to address the issue of monitoring a moving source or scatterer.

A brief discussion of classical parameter resolution bounds and Fisher information is then provided in Sec. VI for measurements obeying the various distributions presented. This is used to show that parameter resolution is highly dependent upon the way that intensity measurements are made. For example, the logarithmic measures commonly used in ocean acoustics, such as scattering strength, target strength and TL, must be derived from a corrected version of the sonar equation that accounts for an inherent bias dependent on the time-bandwidth product of the intensity average. This bias attains its maximum magnitude of 2.5 dB for an instantaneous sample and only vanishes in the deterministic limit of large time-bandwidth product. The logarithmic measures then have mean-square errors that approximate the Cramer–Rao lower bound with increasing accuracy for increasing time-bandwidth product. Finally, a quantitative measure is given for the amount of information that can be lost by certain widely practiced procedures for reducing a set of measurements to a single mean statistic. Such reduction is often employed in ocean-acoustic processing but may be detrimental to subsequent parameter estimates.

I. INTENSITY STATISTICS AS A FUNCTION OF MEASUREMENT TIME AND TEMPORAL COHERENCE

While the expected intensity of a CCGR field may, for all practical purposes, be temporally invariant during a given set of measurements, the experimental measures of intensity are themselves still subject to statistical fluctuation regardless of the amount of finite-time averaging employed. However, under the more stringent assumption of stationarity,[2] many statistical properties of the these fluctuations become invariant and can be readily expressed in terms of the measurement averaging time and the coherence of the received field. As a first step in this direction, the distribution for the average intensity measured from a CCGR field, along with the first few moments, are derived in terms of the time-bandwidth product of the received field. The derivation, presented in Sec. I A, mechanically parallels that given in the optics literature[16] for the statistical properties of polarized thermal light. The way that this analysis extends previously derived ocean-acoustic intensity statistics is discussed in Sec. I B. A brief summary of relevant literature in acoustics, radar, and optics is then given in Sec. I C.

A. The gamma distribution

Let the field measured at a receiver be denoted by $z(t)\exp(-i2\pi f_c t)$. The envelope $z(t)$ contains the stochastic properties of the field modulated at constant carrier frequency f_c. Let both the modulated and demodulated fields be CCGR variables such that the real $x(t)$ and imaginary $y(t)$ components of the envelope $z(t)$ are independent Gaussian random variables with zero mean, and the same variance. Therefore, at any instant t, the probability that the envelope will have value $z=x+iy$ is

$$P(x,y)=\frac{1}{2\pi\langle x^2\rangle}\exp\left\{-\frac{x^2+y^2}{2\langle x^2\rangle}\right\},\quad \text{for}\quad -\infty<x,y<\infty, \tag{1}$$

where the variance of z is given by $\langle x^2\rangle+\langle y^2\rangle$ and $\langle x^2\rangle=\langle y^2\rangle$. Rewriting Eq. (1) in polar coordinates, the uniform phase distribution can be integrated out. The squared magnitude of the field then defines the instantaneous intensity $I(t)=|z(t)|^2$, which obeys an exponential distribution

$$P(I)=\frac{1}{\overline{I}}\exp\left\{-\frac{I}{\overline{I}}\right\},\quad \text{for}\ 0\leqslant I<\infty,$$
$$=0,\quad \text{elsewhere}, \tag{2}$$

with mean $\overline{I}=\langle I\rangle=2\langle x^2\rangle$ and variance $(\overline{I})^2=\langle I^2\rangle-\langle I\rangle^2$. An essential quality of this exponential form is that the most probable value for instantaneous intensity is the same as the most probable value for the field, namely zero.

While the concept of an instantaneous intensity will presently be shown to be more than a mere mathematical stepping stone, it certainly must be used in this way to formulate the statistical properties of actual intensity measurements. This is because actual intensity measurements can only be made over a finite time period but still need to be expressed in terms of the instantaneous fluctuations of an underlying random field. For example, let the average intensity measured over time interval T be

$$W=\frac{1}{T}\int_{-T/2}^{T/2} I(t)\,dt. \tag{3}$$

A measure of the number of independent intensity fluctuations or coherence cells μ averaged during the period T is given by the squared-mean-to-variance ratio or signal-to-noise ratio (SNR) of the measurement W. Under the assumption that the field undergoes stationary fluctuations, such that its autocorrelation $\langle z^*(t)z(t+\tau)\rangle$ is only a function of the time lag τ and not the absolute time reference t, the SNR of W is expressible in terms of the temporal coherence function

$$\gamma(\tau)=\frac{\langle z^*(t)z(t+\tau)\rangle}{\langle |z(t)|^2\rangle}, \tag{4}$$

and the triangle function

$$\begin{aligned}\Delta\left(\frac{\tau}{T}\right)&=1-\left|\frac{\tau}{T}\right|, \quad \text{for } \left|\frac{\tau}{T}\right|\leqslant 1,\\ \Delta\left(\frac{\tau}{T}\right)&=0, \quad \text{elsewhere,}\end{aligned} \tag{5}$$

which arises from auto-convolution of a rectangular measurement window of length T. Specifically, the SNR of W, or equivalently the number of independent intensity fluctuations averaged during the measurement time T, is not restricted to discrete values but rather is defined by the continuous variable

$$\mu=\frac{\langle W\rangle^2}{\langle W^2\rangle-\langle W\rangle^2}=\left[\frac{1}{T}\int_{-\infty}^{\infty}\Delta\left(\frac{\tau}{T}\right)|\gamma(\tau)|^2 d\tau\right]^{-1}, \tag{6}$$

where the last equality is obtained using CCGR moment factoring.[2]

Apart from experimental intrusion, the random field undergoes fluctuations that occur over a characteristic period τ_c referred to as its coherence time scale. A useful measure of the coherence time scale is

$$\tau_c=\int_{-\infty}^{\infty}|\gamma(\tau)|^2 d\tau. \tag{7}$$

Noting the infinite integration limits in this definition, one would expect that more accurate experimental estimates of τ_c would be obtained for longer measurement times T. This is certainly the case, and can be easily shown by considering the limiting form of Eq. (6) for $T\gg\tau_c$. Here, the number of coherence cells is well approximated by the linear relationship $\mu=T/\tau_c$ between the measurement period and coherence time scale. In the opposite extreme of a measurement time much less than the field's intrinsic coherence time scale, such that $T\ll\tau_c$, the integral of Eq. (6) approaches its minimum value of unity, and $\mu=1$ becomes a very good approximation. In this case, the average intensity measurement W is nearly instantaneous, and for practical purposes obeys the exponential distribution of Eq. (2), but not in the vicinity of zero unless μ is identically equal to one, as will be demonstrated presently.

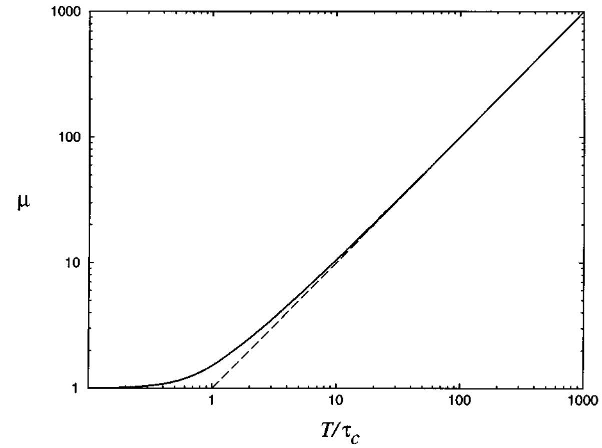

FIG. 1. Time-bandwidth product μ as a function of T/τ_c, where T is the measurement period and τ_c is the intrinsic coherence time scale of the fluctuating field. The solid line shows this dependence when the field has a rectangular frequency spectrum. The dashed line shows the linear asymptote for large T/τ_c.

First, it is useful to consider the general dependence of μ on T/τ_c as illustrated in a particular example. This dependence is given in Fig. 1 for the case in which $z(t)$ has a rectangular frequency spectrum such that $\mathscr{S}(f)=\tau_c$ for $|f|\leqslant 1/(2\tau_c)$ and $\mathscr{S}(f)=0$ elsewhere, $\mathscr{S}(f)$ being the Fourier transform of $\gamma(\tau)$ which takes the form

$$\gamma(\tau)=\frac{\sin(\pi\tau/\tau_c)}{\pi\tau/\tau_c}. \tag{8}$$

It is easy to verify that in an estimate of the spectrum of $z(t)$ over finite time window T, only μ independent frequency components can be resolved by the Rayleigh criteria in the interval $|f|\leqslant 1/(2\tau_c)$, which equals the number of temporal samples attainable at the Nyquist rate.[17] In this case, as in the general case of an integrable spectrum, μ measures the time-bandwidth product of the field received over finite-time window T, just as the ratio μ/T measures its bandwidth. For example, when μ is linearly dependent upon T, as it is for $T\gg\tau_c$, the measured bandwidth μ/T is a good approximation to the intrinsic bandwidth of the fluctuating field $1/\tau_c$. In the opposite extreme when $\tau_c\gg T$, the measured bandwidth of roughly $1/T$ is so dominated by the effect of temporal windowing that it cannot provide useful information about the coherence time scale of the fluctuating field. (A more rigorous discussion of the time-bandwidth product can be found in Ref. 18.)

In this context, if the continuous time average W is replaced by an ensemble average of μ independent and identically distributed instantaneous intensity samples from the same data, both the mean and variance of the resulting ensemble average are identical to those of the original continuous average. Furthermore, the probability distribution for the ensemble average is readily obtained as the inverse Fourier transformation of the characteristic function[2] for instantaneous intensity raised to the power μ. This process leads to the well-known gamma distribution

$$P(W)=\frac{(\mu/\overline{I})^{\mu}W^{\mu-1}\exp\{-\mu(W/\overline{I})\}}{\Gamma(\mu)}, \quad \text{for } 0\leqslant W<\infty,$$
$$=0, \quad \text{elsewhere}, \tag{9}$$

that generally provides an excellent approximation to the exact distribution for averaged intensity W, which has a much less convenient form, as is discussed in detail in Ref. 2. For example, Eq. (9) converges to the exact distribution for W in both the limits of $T\gg\tau_c$ and $T\ll\tau_c$. However, some slight discrepancy between this approximate form and the exact form exists when T is approximately equal to τ_c.

When instantaneous intensity is digitally sampled at the Nyquist rate, as is common practice, Eq. (9) is the exact distribution for discrete-time-averaged intensity. Equation (9), therefore, is essential to the present analysis because it provides a distribution for fluctuating intensity in terms of two quantities that may be estimated with great accuracy and relative ease in an experimental situation, namely the mean intensity $\overline{I}$ and the time-bandwidth product μ of the measurement. Additionally, because the average intensity W has a variance $(\overline{I})^2/\mu$ that depends upon the mean $\overline{I}$, its fluctuations depend on the expected value of the signal and therefore fall under the category of *signal-dependent* noise.

Returning again to the issue of measuring instantaneous intensity in an actual experiment, upon inspection of Eq. (9), it becomes apparent that $\mu=1$ is the only value for μ that does not exclude the possibility of measuring an average intensity W that can be identically zero. For all other values, including μ very near to, but not identically unity, the instantaneous intensity passes through but does not remain at the isolated value of zero during the measurement time. The positive-semidefinite nature of instantaneous intensity then insures that a finite-time average can never yield an average value that is identically zero. The forgoing may explain why, in many experimental situations of high resolution and large sample population, finely binned histograms of fluctuating intensity may follow the exponential distribution very closely except in the vicinity of the origin, where the frequency of samples decays to zero rather than growing to a maximum value.

To investigate the asymptotic properties of the gamma distribution for large time-bandwidth product, it is convenient to employ the general notation

$$m_i(\vartheta)=\langle[\vartheta-m_1(\vartheta)]^i\rangle, \tag{10}$$

for the central moments of integer order $i>1$, where $m_1(\vartheta)=\langle\vartheta\rangle$, and the random variable ϑ obeys an arbitrary probability distribution. It is also convenient to employ the compact notation $\mathscr{G}(m_1(W),\mu)$ to denote the gamma distribution, and $\mathscr{N}(m_1(\eta),m_2(\eta))$ to denote the Gaussian distribution

$$P_{\eta}(\eta)=\frac{1}{[2\pi m_2(\eta)]^{1/2}}\exp\left(-\frac{[\eta-m_1(\eta)]^2}{2m_2(\eta)}\right), \tag{11}$$

where $-\infty<\eta<\infty$.

While it is not difficult, with the aid of a digital computer, to computationally verify that for large μ, the gamma distribution $\mathscr{G}(m_1(W),\mu)$ tends toward a Gaussian $\mathscr{N}(m_1(W),m_2(W))$ with the same mean and variance, an analytic proof has been supplied by Mandel, who demonstrated the asymptotic convergence of all the corresponding moments.[16] A compelling but less complete demonstration can be made by comparison of the kurtosis (peakedness),

$$\mathscr{k}_{\vartheta}=\frac{m_4(\vartheta)}{[m_2(\vartheta)]^2}, \tag{12a}$$

and skew

$$\mathscr{s}_{\vartheta}=\frac{m_3(\vartheta)}{[m_2(\vartheta)]^{3/2}}, \tag{12b}$$

of these distributions, where the necessary moments are readily obtained upon differentiation of the appropriate characteristic functions. For the Gaussian distribution, this procedure leads to a kurtosis $\mathscr{k}_{\eta}=3$, and a skew $\mathscr{s}_{\eta}=0$, the latter indicating symmetry in the distribution of probability about the mean. For the gamma distribution, the kurtosis

$$\mathscr{k}_{W}=3+\frac{6}{\mu}, \tag{13a}$$

and skew

$$\mathscr{s}_{W}=\frac{2}{\sqrt{\mu}}, \tag{13b}$$

have asymptotically Gaussian behavior for $\mu\gg1$, or roughly $\mu\geqslant10$ in practice. However, averaged intensity is positive semidefinite, and therefore its distribution never spans the full domain of a Gaussian, except arguably in the mathematical limit of arbitrarily large μ.

B. Application to ocean-acoustic transmission

As noted during its development, the above formulation for the statistical properties of average intensity is based upon the assumption that the underlying acoustic field at the receiver undergoes CCGR fluctuations. Due to the recurrence of CCGR fields in such a wide variety of otherwise unrelated realizations of stochastic wave propagation, which follows from the central limit theorem, it is often difficult to determine the cause of uncertainty in a particular saturated measurement without soliciting external information. In ocean acoustics, such external information has been historically collected, and a set of basic causes for field fluctuations can be readily cited.

Specifically, the randomization of acoustic fields in the ocean typically arises from (1) incoherent source fluctuation, (2) fluctuation of a scatterer, (3) relative motion between a scatterer and source or receiver in a waveguide (4) relative motion between source and receiver in a waveguide, (5) fluctuation of the waveguide boundary, as for example due to surface waves, (6) medium scintillation or fluctuation in the index of refraction due to such phenomenon as internal waves or turbulence. Typically the last three are associated with transmission scintillation, which becomes saturated after relatively short propagation ranges, greater than a waveguide depth, if many more than a single acoustic mode contributes significantly to the received field, in keeping with the central limit theorem. The origin of the fluctuation in these cases is the change in the interference structure of the waveguide brought on by the given relative motions, which therefore typically need to exceed only a wavelength scale in random amplitude for saturation to occur. For example, Dyer[12] has noted that his model of transmission scintillation ''applies to an ensemble of experiments that contains significant rangewise *motion* of the source with respect to the (multipath interference) structure,'' as in case (4) above. However, it is the point of this paper that such *motion* contributes to the temporal fluctuations whose time scale must be compared to the measurement averaging time to obtain general and accurate statistics. Even early researchers in ocean-acoustics found that these fluctuations can occur over extremely short periods, less than 1 min,[1] so highlighting the need for the more general formulation provided here.

The validity of the CCGR field assumption can easily be tested in practice by examining the statistics of the field received by a hydrophone over time. The extension to previous work lies in the realization that the temporal coherence function or spectrum at the receiver should also be estimated to determine the time-bandwidth product of the measurement. This then leads to the general distribution for averaged intensity as a function of measurement time and temporal coherence given here in Eq. (9).

In this context, the exponential distribution, given in Eq. (9) of Ref. 12 for intensity measured in a ''short-time average,'' is only valid for measurements of instantaneous intensity where $\mu=1$, and not for longer stationary averaging periods for which $\mu>1$. While Dyer defines a ''short-time average'' as ''... an average taken over a time long as compared to $2\pi/\omega$ (the carrier frequency's period) but much less than the stationarity time ...'',[12] this definition is not sufficient to meet the necessary requirement for the exponential intensity distribution to be valid, namely that $\mu=1$. If the ''coherence time'' were substituted for the ''stationarity time'' in this definition, it would be stated correctly in the present context.

The contention of Ref. 12 that the variance of the natural log of averaged intensity is ''independent of any of the metrics of the problem,'' therefore, must be put into perspective. This is because the metric describing the number of coherence cells μ in the intensity average was implicitly assigned to unity. The relevance of these issues to the well known 5.6 dB transmission loss standard deviation of that reference will be examined in Sec. II.

C. Historical notes

Almost immediately after Bergmann's analysis of ocean-acoustic intensity fluctuations for the war effort, and much earlier than its unclassified appearance more than 20 years later, Rice[19] showed that the gamma distribution could be used to describe the statistical properties exhibited by the finite time average of an exponentially distributed random variable. Shortly thereafter, Mandel[16] applied Rice's work in his demonstration that the time-integrated intensity of polarized thermal light undergoes fluctuations that can be well described by the gamma distribution, which has lain the foundation for the formulation given in Sec. I A of the present paper.

With a minor change of variables, the same distribution was obtained independently by Swerling in his case II probability for the radar detection of a fluctuating target by a mean-square processor which integrates over returns from μ active pulses.[3] An ocean-acoustic confirmation of Swerling's work has apparently been given in the underwater target scattering histograms of Dahl and Mathisen.[20] Those histograms were derived by peak amplitude analysis, corresponding to a time-bandwidth product of unity, for which the amplitude distribution is Rayleigh and the intensity distribution is exponential. Similar concepts and distributions have been used to describe the intensity statistics of laser beam fluctuations induced by propagation through the turbulent atmosphere. In that community the model is known as the ''single scatter model'' of atmospheric scintillation.[21,22]

Images derived from CCGR fields exhibit the same kind of signal-dependent noise described by the statistical formulation given in Sec. I A. The noise in such images is commonly referred to as *speckle* when the time-bandwidth product of the measurement is near unity, for which the intensity fluctuations attain their largest variance. Such noise is often found in synthetic aperture radar (SAR), medical ultra sound, side-scan sonar and towed-array reverberation images because the associated active systems typically irradiate terrain of wavelength-scale roughness with narrow-band waveforms of low time-bandwidth product.

II. STATISTICS OF LOGARITHMIC INTENSITY MEASURES AND TRANSMISSION LOSS

It is traditional in ocean-acoustics, and many other disciplines, to measure fluctuating intensity in logarithmic units, although the reasons given for such a measure often seem to verge on scientific folklore. For example, reference is often made[23] to the apparent logarithmic response of human auditory and visual perception to intensity stimulus,[24] the implication being that such a response has been optimized by millions of years of evolution. However, a particularly compelling *quantitative* advantage of the logarithmic measure exists and can be readily cited. That is, for the average intensity of a fully randomized Gaussian field, a logarithmic measure homomorphically transforms signal-dependent noise into additive signal-independent noise. Consequently, optimal methods for finding expected signals or patterns in independent additive noise, which also happen to be well established, can be directly applied when the fluctuating intensity of a CCGR field is measured in logarithmic units.

In this context, a theoretical foundation for logarithmic measures of fluctuating intensity has recently been derived in the optics literature.[25] It is based upon criteria for optimal pattern recognition that follow from statistical estimation, optimal filter and information theory. Because this theory supports the use of traditional logarithmic measures of fluctuating intensity, it should not be surprising that the probability distributions upon which it is based also describe the statistics of ocean-acoustic TL measurements. This formulation is presented here in Sec. II A. The way it extends previous work is discussed in Sec. II B. A brief summary of relevant literature in acoustics and optics is given in Sec. II C.

In a tangential point, the question of whether the theory of Ref. 25 has any bearing on the logarithmic response of human auditory and visual perception to intensity stimulus has been recently raised,[26] primarily because the optical and acoustics fields received by the eye and ear often undergo CCGR fluctuations.

A. The exponential-gamma distribution

The *intensity level* is defined by

$$L = \ln \frac{W}{I_{\mathrm{ref}}}, \tag{14}$$

where I_{ref} is a reference intensity such that the normalized mean intensity is $\overline{I_0} = I/I_{\mathrm{ref}}$. Following Bayes' theorem, the gamma distribution for the measurement W leads to the exponential-gamma distribution

$$P(L) = \frac{(\mu/\overline{I_0})^{\mu}(e^{L})^{\mu-1}\ \exp\{-\mu(e^{L}/\overline{I_0})+L\}}{\Gamma(\mu)}, \quad \text{for } -\infty < L < \infty, \tag{15}$$

for the log-transformed statistic L, which is denoted by $\mathscr{EG}(\langle L\rangle,\mu)$ for notational convenience. The central moments of the exponential-gamma distribution are most readily obtained by direct integration with respect to the variable W. Such a process yields the expected intensity level

$$\langle L\rangle = \ln(\overline{I_0}) + \psi(\mu) - \ln\,\mu, \tag{16a}$$

and variance

$$\langle L^2\rangle - \langle L\rangle^2 = \zeta(2,\mu), \tag{16b}$$

which in general is not inversely proportional to the time-bandwidth product μ. Here, $\psi(\mu)$ is Euler's psi function[27] and $\zeta(2,\mu)$ is Riemann's zeta function.[27] Formulas defining these special functions are given in Appendix A. For example, $\zeta(2,1) = \pi^2/6$, and in the limit as $\mu \gg 1$, the expectation value of L converges to $\ln(I_0)$ while the variance $\zeta(2,\mu)$ approaches $1/\mu$.

It is of great practical significance that intensity level L has a variance that does not depend on the expected intensity I, as average intensity W does, but only on the time-bandwidth product μ of the measurement. For example, suppose that the instantaneous intensity of a CCGR field is practically stationary over short periods, but not over long periods where some trend in expected intensity emerges. Further suppose that this trend is to be measured by time series analysis, where average intensity samples are consecutively collected over short-term stationary periods and then concatenated. The resulting intensity time series will have a standard deviation that is directly proportional to the local value of expected intensity, making comparison of samples with different expectation values difficult. Conversion to logarithmic units, however, homomorphically transforms such signal-dependent noise into additive signal-independent noise. The log-transformed time series has a uniform standard deviation when the time-bandwidth product of the measurement is chosen to be constant for all samples. Well-established techniques for processing signals in additive noise are then appropriate. In particular, the optimal method for recognizing the nonstationary trend is provided by matched filtering the time series with hypothetical trends in the log-transformed domain.[25] This technique has also proven to be valuable for recognizing patterns in active sonar images.[14,28] The *speckle* found in such images, which is most pronounced for low time-bandwidth product measures, arises from the same CCGR field fluctuations described here. Characteristic scales of the nonstationary trends in both saturated intensity time series and speckled images are best measured in the logarithmic domain.[25]

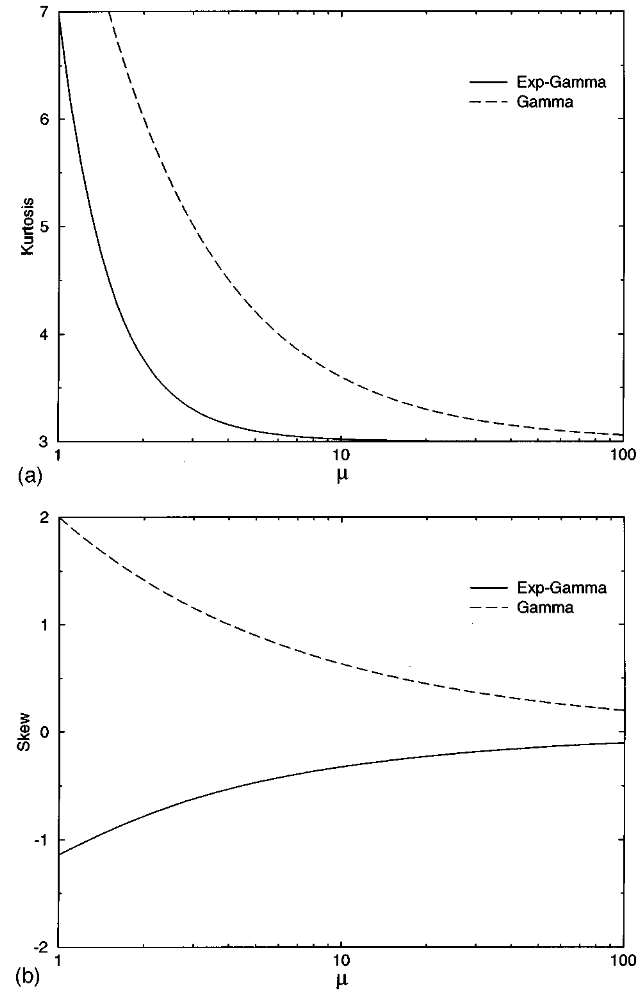

FIG. 2. (a) Kurtosis for the exponential-gamma distribution in solid line and for the gamma distribution in dashed line. The Gaussian kurtosis is 3. (b) Skew for the exponential-gamma distribution in solid line and for the gamma distribution in dashed line. The Gaussian skew is zero.

Some of the asymptotic properties of the exponential-gamma distribution can be ascertained by considering its kurtosis

$$\mathscr{k}_L = 3 + 6\,\frac{\zeta(4,\mu)}{[\zeta(2,\mu)]^2}, \tag{17a}$$

and skew

$$\mathscr{s}_L = -2\,\frac{\zeta(3,\mu)}{[\zeta(2,\mu)]^{3/2}}, \tag{17b}$$

for increasing μ. Apparently, the exponential-gamma approaches a Gaussian with the same mean and variance more rapidly than the gamma distribution, as is evident after inspection of the curves provided in Fig. 2. For practical purposes, the criterion that $\mu>4$ appears to be sufficient for $\mathscr{EG}(\langle L\rangle,\mu)$ to be approximated by $\mathscr{N}(\langle L\rangle,\langle L^2\rangle-\langle L\rangle^2)$.

The Gaussian asymptote of the exponential-gamma distribution can also be obtained by analytic means. After some straight forward algebraic manipulation, Eq. (15) can be rewritten as

$$P(L)=\left(\frac{\mu^\mu}{\Gamma(\mu)}\right)\exp(\mu\{[L-\ln(\overline{I_0})]-\exp[L-\ln(\overline{I_0})]\}). \tag{18}$$

The argument of the interior exponential can then be expanded into a Taylor series such that

$$P(L)\approx\left(\frac{\mu^\mu}{\Gamma(\mu)}\exp(-\mu)\right)\exp\left(-\frac{\mu}{2}\,|L-\ln(\overline{I_0})|^2\right), \tag{19}$$

for $|L-\ln(\overline{I_0})|\ll 3$.[25,29] This can be used as a relatively crude approximation even for $\mu=1$ because the corresponding standard deviation of L is then $\pi/\sqrt{6}=1.3$. As μ increases the approximation becomes better. By use of Stirling's formula[29] for $\mu\gg1$, Eq. (19) transforms to the Gaussian

$$P(L)\approx\left(\frac{\mu}{2\pi}\right)^{1/2}\exp\left(-\frac{\mu}{2}\,|L-\ln(\overline{I_0})|^2\right), \tag{20}$$

so that the exponential-gamma distribution for the intensity level L is distributed according to $\mathscr{N}(\ln(\overline{I_0}),1/\mu)$ for large μ. Therefore, as the time-bandwidth product of the measurement becomes large, the expected intensity level approaches the logarithm of expected intensity, the intensity level variance approaches the inverse time-bandwidth product, and the exponential-gamma distribution converges to a Gaussian.

B. Application to ocean-acoustic transmission loss statistics

Due to its simple linear dependence on intensity level L, transmission loss H, sampled after saturated ocean-acoustic propagation, has statistical properties that can be readily expressed in terms of the time-bandwidth product of the measurement. Specifically, TL is related to intensity level by the equation

$$H=-10\log(e^L)+K, \tag{21}$$

where H is measured in dB *re*: 1 m, and K is a conversion factor in dB *re*: 1 μPa and m that can be set to zero without any loss of generality. Therefore, the probability distribution for H is readily found to be exponential-gamma, so that the mean TL

$$\langle H\rangle=-(10\log e)[\ln(\overline{I_0})+\psi(\mu)-\ln\mu], \tag{22}$$

not only depends on the mean intensity $\overline{I}$, but also on the time-bandwidth product of the measurement. The TL standard deviation

$$\langle H^2\rangle-\langle H\rangle^2=(10\log e)\mathrm{sqrt}[\zeta(2,\mu)], \tag{23}$$

on the other hand, only depends upon the time-bandwidth product. When these moments are plotted as a function of μ, as in Fig. 3, it becomes evident that Dyer's 5.6 dB TL standard deviation and 2.5-dB augmentative bias in mean TL are only valid for an instantaneous measurement, for which $\mu=1$, and not for longer stationary averages as might be inferred from Ref. 12. As the time-bandwidth product increases, the TL standard deviation asymptotically approaches zero along the curve $(10\log e)\mathrm{sqrt}[1/\mu]$, or roughly $4.34\mathrm{sqrt}[1/\mu]$, while the TL mean approaches $-10\log$ (the mean-square transmission) as it must in the deterministic limit of an arbitrarily large sample size. Additionally, the asymptotic analysis of the previous section shows that TL statistics in the saturated region can legitimately be described as normal for measurement time-bandwidth products exceeding four, which runs counter to some previous suggestions implicitly based upon unity time-bandwidth measures.[12]

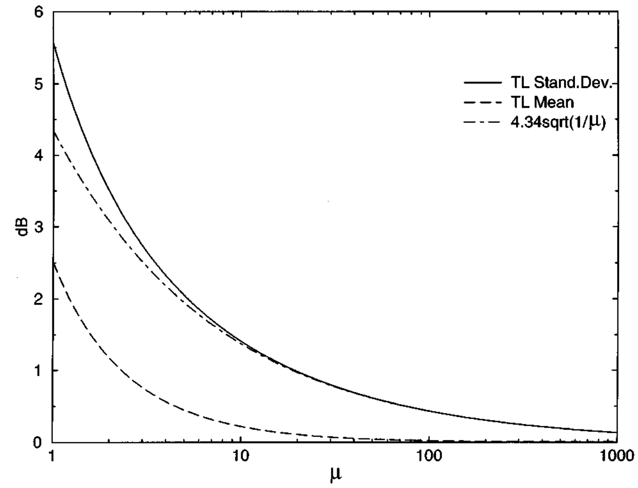

FIG. 3. Transmission loss (TL) standard deviation sqrt$[\langle H^2\rangle-\langle H\rangle^2]$ in solid line and TL mean $\langle H\rangle$ for $\overline{I_0}=1$ in dashed line as a function of time-bandwidth product μ. The asymptotic dependence of the TL standard deviation for large μ, $(10\log e)\mathrm{sqrt}[1/\mu]$, is also given in dot-dashed line.

C. Historical notes

What is referred to here as the exponential-gamma distribution was apparently first derived by Dyer[12] for an entirely different purpose, namely to describe the ''noise of multiple distant sources.'' However, the relationship between degrees of freedom, integration time and temporal coherence was not addressed, as is discussed further in the next section. The distribution was later rediscovered by Barakat[4] in an analysis of laser beam speckle patterns observed through a finite spatial aperture. In Barakat's analysis, the relationship between degrees of freedom, spatial aperture and spatial coherence is derived in accord with Goodman.[30] It is noteworthy that Pierce has arrived at a decibel standard deviation equivalent to $4.34\mathrm{sqrt}(1/\mu)$ dB for Gaussian random acoustic signals without exploiting the convenience of complex variables.[31] The derivation of the Gaussian asymptote of the exponential-gamma distribution follows the analysis of Arsenault and April[29] and Makris.[25]

III. THE AVERAGE INTENSITY OF MULTIPLE INDEPENDENT CIRCULAR COMPLEX GAUSSIAN RANDOM FIELDS

When multiple independent CCGR fields are superposed at a receiver, fluctuations in average intensity can once again be described by the gamma distribution, as is shown in Sec. III A. The way that this extends and generalizes previous statistical formulations for the ''noise of multiple distant sources'' and a ''signal plus noise'' is discussed in Sec. III B.

A. The gamma distribution

When multiple independent *fields* are superposed at a receiver, their individual variances, or equivalently their expected intensities, sum to the variance of the total field, which constitutes the expected intensity at the receiver. However, the *intensities* of the independent fields may not comprise mutually independent statistics of the measurement except under certain special circumstances.

To illustrate this situation, let the total received field $z(t)\exp(-i2\pi f_c t)$ be the sum of S independent CCGR fields $z_j(t)\exp(-i2\pi f_c t)$, emanating from distinct sources or scatterers, modulated at carrier frequency f_c such that

$$z(t)=\sum_{i=1}^{S} z_i(t). \tag{24}$$

Then it is readily verified with characteristic functions that $z(t)$ is a CCGR variable with variance $\overline{I}=\langle|z^2|\rangle$ equal to the sum of the variances of the $z_i(t)$, which are denoted by $\overline{I_i}=\langle|z_i^2|\rangle$. The expected intensity of the superposed fields is then

$$\overline{I}=\sum_{i=1}^{S} \overline{I_i}. \tag{25}$$

While the average intensity W of the superposed fields, measured during period T, still obeys the gamma distribution $\mathscr{G}(I,\mu)$, the time-bandwidth product μ, as defined in Eq. (6), is now dependent upon the temporal coherence function of the superposed fields. Furthermore, when the variance of W, namely $(\overline{I})^2/\mu$, is expanded in terms of the intensities of the component fields, cross terms with factors $\overline{I_i}\,\overline{I_j}$ emerge. These cross terms appear because the instantaneous intensities of the constituent fields are not independently superposed in an instantaneous measurement of the total field intensity. An alternative proof of this is given in Appendix B.

Suppose now that the desired signal $z_1(t)$, with intensity I_1, is just one component among many received in a noisy environment, where the noise field is $z(t)-z_1(t)$, with intensity $\overline{I_N}=\overline{I}-\overline{I_1}$. Given the average intensity measurement W, the SNR for the intended signal component then becomes $\mu(\overline{I_1}/\overline{I})^2$, which includes the effects of both additive noise and signal-dependent field fluctuations. Here, the time-bandwidth product μ depends upon the coherence function of the combined signal and noise fields. Consequently, if the additive noise extends over a much broader frequency band than the signal, stationary averaging will reduce the additive noise component of the variance of W far more rapidly than the signal-dependent component. A more effective way of eliminating out-of-band noise is to filter the total received field to the signal band, as is typically accomplished with a matched filter.

B. Application to ocean-acoustic transmission statistics

The distributions analyzed in Sec. III A of this paper, namely Eqs. (9) and (15) with I defined for multiple sources, also characterize the ''noise of multiple distant sources'' studied by Dyer.[12] However, there are substantial differences between Dyer's formulation and that given here. Specifically, the distributions defined in Eqs. (27) and (28) of Ref. 12 were derived under the assumption that the intensity measured from each distant source is independent, exponentially distributed and comprised of a single tone spectrally disjoint from the simultaneously measured tones of the other sources. However, the condition of Ref. 12 that each source be of ''different frequency'' is *not sufficient* to insure that the intensity contribution measured from any one source will be independent of that measured from any other. The *additional requirement* for such independence is that the inverse measurement time $1/T$ must be less than the spectral separation between any of the contributing fields. Further, the intrinsic bandwidth of the field received from each source must be far less than $1/T$ for the measured contribution from that source to be virtually instantaneous, as is necessary for its intensity distribution to be approximately exponential. These requirements can be stated mathematically by expressing the time-bandwidth product in terms of the spectrum $\mathscr{S}(f)$ of the total received field

$$\mu=\left[\frac{1}{T^2}\int_{-\infty}^{\infty}\int \mathscr{S}(f)\,\mathscr{S}^*(f')\left(\frac{\sin(\pi T(f-f'))}{\pi(f-f')}\right)^2 df'\,df\right]^{-1}, \tag{26}$$

which is simply another representation of Eq. (6). The first integration is a convolution which sets the number of independent frequency cells, while the second integration sums these cells. The cells are measured in units of $1/T$, the distance from the apex of the sinc function to its first zero crossing, in accord with the Rayleigh Criterion. Therefore, the previous statistical description of ''the noise of multiple distant sources,'' based upon the assumption that the *intensities* received simultaneously from the noise sources are independent,[12] comprises a special case of the more general formulation provided here, where it is assumed that the *fields* received from the noise sources are independent.

The distributions for ''signal plus noise'' of the present section differ substantially from those given in Eqs. (33) and (35) of Ref. 12 for reasons similar to those just discussed. Here, for example, TL for a ''signal plus noise'' field obeys the exponential-gamma distribution with standard deviation $(10\log e)\mathrm{sqrt}[\zeta(2,\mu)]$ dB, where the time-bandwidth product μ is for a measurement of the combined fields with expected intensity $\overline{I}=\overline{I_1}+\overline{I_N}$. Therefore, the TL standard deviation for a ''signal plus noise'' field never exceeds 5.6 dB as it may in Ref. 12.

IV. THE AVERAGE OF INDEPENDENT GAMMA-DISTRIBUTED INTENSITY MEASUREMENTS

In ocean-acoustics, there are many situations in which independent and presumably gamma-distributed intensity measurements are averaged. For example, in displaying the beamformed output of a hydrophone array, it is common practice to reduce the variance by averaging the uncorrelated intensities received on adjacent nonoverlapping beams.[14] Similarly, it is sometimes convenient to average independent multipath arrivals that are temporally disjoint,[14] or to average independent measurements of backscatter to reduce the variance in scattering strength or target strength estimation.[15] To investigate the statistical properties of such measurements, a distribution is derived for the average of two independent gamma-distributed intensity samples that may not be identically distributed. The asymptotic Gaussian form of this distribution is then generalized to describe the average of an arbitrary number of independent but not necessarily identically distributed intensity samples. This form is later used, in Sec. VI, to quantify the maximum amount of information that can be inferred about a desired parameter set from an amalgamated measurement.

Let the average intensities W_α and W_β be independent and, respectively, obey the nonidentical gamma distributions $\mathscr{G}(I_\alpha,\mu_\alpha)$ and $\mathscr{G}(I_\beta,\mu_\beta)$. The distribution for the sum $\kappa=W_\alpha+W_\beta$ is readily obtained as the inverse Fourier transform of the product of the characteristic functions of W_α and W_β. With the aid of tabulated integral transforms,[27,32] the distribution for κ is found to be

$$P(\kappa)=\frac{(\kappa\mu_\alpha/\overline{I_\alpha})^{\mu_\alpha}(\kappa\mu_\beta/\overline{I_\beta})^{\mu_\beta}}{\kappa\Gamma(\mu_\alpha+\mu_\beta)}\exp\left\{-\frac{\mu_\beta}{\overline{I_\beta}}\kappa\right\}\times{}_1F_1\left[\mu_\alpha;\mu_\alpha+\mu_\beta;\kappa\left(\frac{\mu_\beta}{\overline{I_\beta}}-\frac{\mu_\alpha}{\overline{I_\alpha}}\right)\right],\quad \text{for}\quad \kappa\geqslant 0, \tag{27}$$

and zero elsewhere, where ${}_1F_1$ is Kummer's confluent hypergeometric series.[32] The mean of κ is $\langle\kappa\rangle=I_\alpha+I_\beta$, and the variance is

$$\langle\kappa^2\rangle-\langle\kappa\rangle^2=\frac{(\overline{I_\alpha})^2}{\mu_\alpha}+\frac{(\overline{I_\beta})^2}{\mu_\beta}. \tag{28}$$

It is often possible to make the simplification that the time-bandwidth product of the two measurements is equal so that $\mu_\alpha=\mu_\beta=\mu$. The distribution for the sum κ, readily obtained by convolution of the distributions for W_α and W_β, is then

$$P(\kappa)=\frac{\sqrt{\pi}}{\Gamma(\mu)}\left(\frac{\mu\kappa}{\overline{I_\alpha}-\overline{I_\beta}}\right)^{\mu}\left(\frac{\mu}{\kappa}\frac{(\overline{I_\alpha}-\overline{I_\beta})}{\overline{I_\alpha}\overline{I_\beta}}\right)^{1/2}\times\exp\left\{-\frac{\mu\kappa}{2}\frac{(\overline{I_\alpha}+\overline{I_\beta})}{\overline{I_\alpha}\overline{I_\beta}}\right\}I_{\mu-1/2}\left(\frac{\mu\kappa}{2}\frac{(\overline{I_\alpha}-\overline{I_\beta})}{\overline{I_\alpha}\overline{I_\beta}}\right),\quad\text{for}\quad\kappa\geqslant 0, \tag{29}$$

and zero elsewhere, where $I_{\mu-1/2}$ is a modified Bessel function of the first kind of order $\mu-1/2$. It is noteworthy that, with a minor change of variables, this equation describes the probability distribution for partially coherent thermal light.[2,33] The mean of κ is still $\langle\kappa\rangle=I_\alpha+I_\beta$ while the variance simplifies to

$$\langle\kappa^2\rangle-\langle\kappa\rangle^2=\frac{(\overline{I_\alpha})^2+(\overline{I_\beta})^2}{\mu}. \tag{30}$$

The kurtosis

$$\mathscr{k}_\kappa=3+\left(\frac{6}{\mu}\right)\frac{(\overline{I_\alpha})^4+(\overline{I_\beta})^4}{[(\overline{I_\alpha})^2+(\overline{I_\beta})^2]^2}, \tag{31a}$$

and skew

$$\mathscr{s}_\kappa=\left(\frac{2}{\sqrt{\mu}}\right)\frac{(\overline{I_\alpha})^3+(\overline{I_\beta})^3}{[(I_\alpha)^2+(I_\beta)^2]^{3/2}}, \tag{31b}$$

indicate that Eq. (29) converges to the Gaussian $\mathscr{N}(\langle\kappa\rangle,\langle\kappa^2\rangle-\langle\kappa\rangle^2)$ for $\mu\gg 1$, as a natural consequence of the central limit theorem.

When it is the average of the two measurements that is of interest, the random variable κ must be redefined as $\kappa=(W_\alpha+W_\beta)/2$. This average is distributed according to Eq. (29) but with I_α replaced by $I_\alpha/2$ and I_β by $I_\beta/2$. Assuming the statistics across the two measurements are independent and ergodic, such that $I=I_\alpha=I_\beta$, the average κ obeys the distribution $\mathscr{G}(I,2\mu)$, as is consistent with a doubling of the time-bandwidth product of a constituent sample. By asymptotic analysis of Eq. (29), a rigorous demonstration of this, and the case when I_β approaches zero, is provided in Appendix C. Similarly, for an ergodic population of independent samples distributed according to $\mathscr{G}(I,\mu)$, the average of D samples of this population obeys $\mathscr{G}(I,D\mu)$.

The more general scenario is the ensemble average of D independent intensity measurements W_i that, respectively, obey the nonidentical gamma distributions $\mathscr{G}(\langle W_i\rangle,\mu_i)$. Let this be described by

$$\kappa=\frac{1}{D}\sum_{i=1}^{D}W_i. \tag{32}$$

For $\mu_i\gg 1$, the probability distribution for κ converges to the Gaussian $\mathscr{N}(\langle\kappa\rangle,\langle\kappa\rangle^2-\langle\kappa\rangle^2)$, where respective linear summations of the constituent means and variances lead to the mean

$$\langle\kappa\rangle=\frac{1}{D}\sum_{i=1}^{D}\langle W_i\rangle, \tag{33a}$$

and variance

$$\langle\kappa^2\rangle-\langle\kappa\rangle^2=\frac{1}{D^2}\sum_{i=1}^{D}\frac{\langle W_i\rangle^2}{\mu_i}, \tag{33b}$$

of the average, as follows from the independence of the constituent measurements. If now each W_i represents the average intensity of S_i independent CCGR fields, the expectation value for each W_i is

$$\langle W_i\rangle=\sum_{j=1}^{S_i}\overline{I_{ij}}, \tag{34}$$

in accord with the analysis of the previous section. The utility of this representation will become evident in Sec. VI and Appendix E, where a quantitative measure of the amount of information that can be lost by such an averaging process, common in ocean-acoustic applications, is presented.

V. THE RATE OF CHANGE OF AVERAGE INTENSITY

It is sometimes possible to infer the velocity of a moving source or scatterer from a series of incoherent intensity measurements or images.[34] When this is possible, it is necessary to compute the time rate of change of measured intensity by the method of finite differences. While such a velocity estimate may also require computation of finite differences in space to account for advection in the image plane, the distinction between temporal and spatial domain is not crucial to the present statistical analysis.

Suppose two consecutive averaged intensity measurements W_α and W_β are separated by a time interval Δt long enough to insure that they are independent. Let these measurements obey respective nonidentical gamma distributions $\mathscr{G}(I_\alpha,\mu)$ and $\mathscr{G}(I_\beta,\mu)$. The random variable for intensity rate is then defined by the difference quotient $V=(W_\beta-W_\alpha)/\Delta t$. Its probability distribution is readily obtained by inverse Fourier transformation of the relevant characteristic functions. With the aid of the tabulated integral transforms,[27] the distribution for V is found to be

$$\begin{aligned}P(V)=&\frac{1}{\Gamma(\mu)}\left(\frac{\mu|V|\Delta t}{\overline{\overline{I_\alpha+I_\beta}}}\right)^{\mu}\left(\frac{\mu\Delta t}{\pi|V|}\,\frac{(\overline{I_\alpha}+\overline{I_\beta})}{\overline{\overline{I_\alpha I_\beta}}}\right)^{1/2}\\&\times\exp\left\{-\frac{\mu\Delta t}{2}\,\frac{|V(\overline{I_\beta}-\overline{I_\alpha})|}{\overline{\overline{I_\alpha I_\beta}}}\right\}\\&\times K_{1/2-\mu}\left(\frac{\mu|V|\Delta t}{2}\,\frac{(\overline{I_\alpha}+\overline{I_\beta})}{\overline{\overline{I_\alpha I_\beta}}}\right)\\&\text{for }\;-\infty<V<\infty,\end{aligned}\tag{35}$$

where $K_{1/2-\mu}$ is a modified Bessel function of the second kind of order $1/2-\mu$. The mean is $\langle V\rangle=(I_\beta-I_\alpha)/\Delta t$ and the variance is

$$\langle V^2\rangle-\langle V\rangle^2=\frac{(\overline{I_\alpha})^2+(\overline{I_\beta})^2}{\mu(\Delta t)^2}.\tag{36}$$

The kurtosis $\mathscr{k}_V$, given by the right-hand side of Eq. (31a), and skew

$$\mathscr{s}_V=\left(\frac{2}{\sqrt{\mu}}\right)\frac{(\overline{I_\beta})^3-(\overline{I_\alpha})^3}{[(I_\alpha)^2+(I_\beta)^2]^{3/2}},\tag{37}$$

indicate that Eq. (35) converges to the Gaussian $\mathscr{N}(\langle V\rangle,\langle V^2\rangle-\langle V\rangle^2)$ for $\mu\gg1$, in accord with the central limit theorem.

To obtain the second time derivative of intensity, A, three measurements are needed. If these are equally spaced in time, the distribution of

$$A=\frac{W_\alpha-2W_\beta+W_\gamma}{(\Delta t)^2},\tag{38}$$

converges to the Gaussian $\mathscr{N}(\langle A\rangle,\langle A^2\rangle-\langle A\rangle^2)$ for $\mu\gg1$, with mean

$$\langle A\rangle=\frac{\overline{I_\alpha}-2\overline{I_\beta}+\overline{I_\gamma}}{(\Delta t)^2},\tag{39a}$$

and variance

$$\langle A^2\rangle-\langle A\rangle^2=\frac{(\overline{I_\alpha})^2+(2\overline{I_\beta})^2+(\overline{I_\gamma})^2}{\mu(\Delta t)^4}.\tag{39b}$$

To compute higher-order derivatives, well-known finite difference equations can be used.[35] However, the variance is proportional to the weighted sum of the variances of the constituent measures, and gains positive terms linearly with the order of the derivative to be estimated. Therefore, for fixed μ, the SNR of the derivative estimate typically decreases as the order of the derivative increases.

VI. THE RESOLUTION OF PARAMETERS

Parameter resolution is highly dependent upon the way that intensity measurements are made. To show this, classical parameter resolution bounds and Fisher information matrices are derived for the various kinds of measurements analyzed in previous sections. In particular, the logarithmic measures commonly used in ocean acoustics, such as scattering strength, target strength and TL, must be derived from a corrected version of the sonar equation that accounts for an inherent bias dependent on the time-bandwidth product of the intensity average. The logarithmic measures then have mean-square errors that approximate the Cramer–Rao lower bound (CRLB) with increasing accuracy for increasing time-bandwidth product. Additionally, it is shown how information can be lost by certain widely practiced procedures for reducing a set of independent samples to a single mean statistic, as is often done in ocean-acoustic processing.

Suppose that a general N_a-dimensional parameter vector $\mathbf{a}$ is to be estimated from the N-dimensional measurement vector $\mathbf{Y}$. According to estimation theory,[36] the mean-square error of any unbiased estimate $\hat{a}_i$, based upon measurement vector $\mathbf{Y}$, can never be less than the CRLB[36]

$$E[(\hat{a}_i-a_i)^2]\geq[\mathbf{J}^{-1}(\mathbf{a})]_{ii},\tag{40}$$

where a_i is the true parameter value and $\mathbf{J}(\mathbf{a})$ is the N_a by N_a Fisher information matrix[36] with elements

$$J_{ij}(\mathbf{a})=-E\left[\frac{\partial^2}{\partial a_i\,\partial a_j}\ln P(\mathbf{Y}|\mathbf{a})\right].\tag{41}$$

A. Information in intensity and log-transformed intensity measures

Suppose the parameters $\mathbf{a}$ are to be estimated from a set of N independent gamma-distributed intensity measurements W_k contained in the vector $\mathbf{W}$. Because no information is lost in the homomorphic transformations $L_k=\ln(W_k/I_{\mathrm{ref}})$, where the L_k comprise the vector $\mathbf{L}$, the Fisher information matrices for measurements $\mathbf{W}$ and $\mathbf{L}$ are identical and equal to[25]

$$J_{ij}(\mathbf{a})=\sum_{k=1}^{N}\frac{\mu_k}{(\overline{I_k}(\mathbf{a}))^2}\,\frac{\partial\overline{I_k}(\mathbf{a})}{\partial a_i}\,\frac{\partial\overline{I_k}(\mathbf{a})}{\partial a_j},\tag{42}$$

as shown in Appendix D. This expression exhibits behavior that should be expected in a measure of information. Namely, information is positive semidefinite, and the information contained in a joint set of independent measurements equals the cumulative information of all the individual measurements. Consequently, the CRLB on parameter resolution is only singular when the Fisher information for all joint measurements is zero, as is evident by inspection of the particular case given in Eq. (42).

Although both intensity measurements and their log-transforms contain the same information, the information about parameters linearly related to their respective expectation values is substantially different. For example, given the single gamma-distributed measurement W, the CRLB for estimation of $\overline{I}$ is simply the variance of the measurement $J^{-1}(\overline{I})=(\overline{I})^2/\mu$, so that the measurement W itself attains the CRLB as an unbiased estimate of $\overline{I}$.

The situation, however, is different for logarithmic measures. By straightforward manipulation of Eq. (42), the Fisher information for the measurements **W** or **L** can be rewritten as[25]

$$J_{ij}(\mathbf{a})=\sum_{k=1}^{N}\mu_k\frac{\partial\langle L_k\rangle}{\partial a_i}\frac{\partial\langle L_k\rangle}{\partial a_j},\tag{43}$$

with the understanding that the μ_k do not depend on **a**. Given the single exponential-gamma-distributed measurement L, the CRLB for estimation of $\langle L\rangle$ therefore is $J^{-1}(\langle L\rangle)=1/\mu$. While the variance of L is approximately equal to this bound even for $\mu=1$, it only converges to it in the asymptotic limit of large μ, as shown in Sec. II. The CRLB, therefore, can only be attained asymptotically for parameters linearly related to the expected value of intensity level. One should realize, however, that for practical purposes such asymptotic convergence occurs for relatively small μ, as noted in Sec. II.

B. The corrected sonar equation

If the estimate, based upon the exponential-gamma-distributed measurement L, is for the unknown decibel quantity

$$a=(10\log e)[\ln\overline{I_0}+\Omega_0],\tag{44}$$

as it often is in practice, where Ω_0 is some known constant, the unbiased estimator

$$\hat{a}=(10\log e)[L+\Omega_0-\psi(\mu)+\ln\mu],\tag{45}$$

only converges to $(10\log e)[L+\Omega_0]$ in the asymptotic limit of large μ, where it also attains the CRLB on root-mean-square error $J^{-1/2}(a)=(10\log e)\mathrm{sqrt}[1/\mu]$. The more frequently encountered estimator $(10\log e)[L+\Omega_0]$, however, has a negative bias that decays from -2.5 dB at $\mu=1$ to zero along the curve $(10\log e)[\psi(\mu)-\ln\mu]$ as the time-bandwidth product μ increases. Misguided use of this biased estimator in reverberation analysis, for example, can lead one to draw the erroneous conclusion that scattering strength increases with the time-bandwidth product of the measurement.

The familiar sonar equation $\hat{a}=(10\log e)[L+\Omega_0]$, therefore, is only valid in the deterministic limit of large μ. The corrected sonar equation given in Eq. (45) includes the additional terms necessary to account for the μ-dependent bias introduced by an exponential-gamma-distributed intensity-level measurement and is therefore the more general form.

C. Information in the average intensity of superposed fields

The situation becomes more complex when each W_k measures the average intensity of a superposition of S_k independent CCGR fields, as in Sec. III where reception of the ''noise of multiple distant sources'' and a ''signal plus noise'' is considered. In this case, the Fisher information matrix of Eq. (42) becomes

$$J_{ij}(\mathbf{a})=\sum_{k=1}^{N}\frac{\mu_k\Sigma_{q=1}^{S_k}\Sigma_{r=1}^{S_k}\dfrac{\partial\overline{I_{qk}}(\mathbf{a})}{\partial a_i}\dfrac{\partial\overline{I_{rk}}(\mathbf{a})}{\partial a_j}}{[\Sigma_{l=1}^{S_k}\overline{I_{lk}}(\mathbf{a})]^2}.\tag{46}$$

This form is useful because it explicitly shows how information partitioned among the component fields is incorporated in an intensity measurement of their combination. Additionally, it should be understood that in such a measurement, some information carried by each individual field is lost. This is shown explicitly in Appendix E.

D. Information in amalgamated intensity averages

Fisher information for measurements obeying more complicated distributions, such as those described in Secs. IV and V, is most readily obtained by the methodology of Ref. 37. Specifically, suppose the parameters **a** only depend on the expectation values $\mathbf{M}=\langle\mathbf{Y}\rangle$. Then, given expressions for **M** and $\mathbf{J}(\mathbf{M})$, the Fisher information matrix $\mathbf{J}(\mathbf{a})$ can be obtained from the equation[38]

$$J_{ij}(\mathbf{a})=\frac{\partial\mathbf{M}^T}{\partial a_i}\mathbf{J}(\mathbf{M})\frac{\partial\mathbf{M}}{\partial a_j}.\tag{47}$$

Along these lines, consider the amalgamated average κ of D independent gamma-distributed measurements W_l that each, respectively, sample the intensity of S_l independent superposed CCGR fields, as described in Sec. IV. For the joint measurements of N such independent samples κ_k, the Fisher information matrix is given by

$$J_{ij}(\mathbf{a})=\sum_{k=1}^{N}\frac{\Sigma_{q=1}^{D_k}\Sigma_{r=1}^{S_{qk}}\Sigma_{m=1}^{D_k}\Sigma_{n=1}^{S_{mk}}\dfrac{\partial\overline{I_{qrk}}(\mathbf{a})}{\partial a_i}\dfrac{\partial\overline{I_{mnk}}(\mathbf{a})}{\partial a_j}}{\Sigma_{l=1}^{D_k}\dfrac{[\Sigma_{t=1}^{S_{lk}}\overline{I_{ltk}}(\mathbf{a})]^2}{\mu_{lk}}}.\tag{48}$$

While this expression is certainly cumbersome, it is not without practical value. In particular, Eq. (48) not only quantifies the information contained in the kind of amalgamated intensity measurements that are widely used in ocean acoustics, but also expresses this information in terms of constants of both the measurement process and constituent fields. The amount of information lost in such an amalgamation, however, can be significant, as is shown in Appendix E.

For illustrative purposes, consider the case when only a single parameter is to be estimated from a single measurement, and that parameter is either the expectation value of the measurement or is linearly related to it. Given the amalgamated measurement κ, the parameter $\overline{I_{11}}$, representing the expected intensity of a single field component of the respective intensity sample W_1, cannot be resolved better than

$$J^{-1}(\overline{I_{11}})=\sum_{l=1}^{D}\frac{[\Sigma_{t=1}^{S_l}\overline{I_{lt}}(\mathbf{a})]^2}{\mu_l}. \tag{49}$$

This bound equals $D^2J^{-1}(\langle\kappa\rangle)$, and therefore is proportional to, but potentially much greater than the variance of the amalgamated measurement κ. If such amalgamation cannot be avoided, accurate estimation of the intensity $\overline{I_{11}}$ requires the time-bandwidth products of the constituent gamma-distributed intensity samples to be sufficiently large that $\overline{I_{11}}$ greatly exceeds sqrt$[J^{-1}(\overline{I_{11}})]$.

In many practical scenarios in ocean-acoustics, however, such amalgamation cannot be avoided. Consider, for example, a problem encountered in remotely imaging the ocean basin with an active towed-array system.[14] Returns from shadow-zone sites, which lie between convergence zones, generally arrive in time so that no particular path to the seafloor makes a dominant contribution during a given measurement period. These returns may also arrive simultaneously and consecutively during the measurement, leading to an amalgamated average of the intensities of superposed and temporally disjoint fields. In such cases, it is often difficult to obtain sufficiently large time-bandwidth products to resolve the mean contribution of a particular seafloor patch. Returns from shadow zone ranges can then be adequately described as *clutter* because they hinder presently known means of inferring geomorphological features of the ocean basin.

E. Information in intensity-rate measures

Finally, consider the intensity rate measurement $V=(W_\beta-W_\alpha)/\Delta t$, of Sec. V, where both W_α and W_β are gamma-distributed. By application of Eq. (47), the Fisher information matrix for N independent measurements of intensity rate V_k is

$$J_{ij}(\mathbf{a})=\sum_{k=1}^{N}\frac{\mu_k\left(\frac{\partial\overline{I_{\beta_k}}(\mathbf{a})}{\partial a_i}-\frac{\partial\overline{I_{\alpha_k}}(\mathbf{a})}{\partial a_i}\right)\left(\frac{\partial\overline{I_{\beta_k}}(\mathbf{a})}{\partial a_j}-\frac{\partial\overline{I_{\alpha_k}}(\mathbf{a})}{\partial a_j}\right)}{[\overline{I_{\alpha_k}}(\mathbf{a})]^2+[\overline{I_{\beta_k}}(\mathbf{a})]^2}. \tag{50}$$

Evidentially, the information about some parameter a_i contained in an intensity-rate measurement V_k may vanish when the expectation value of V_k is zero even if the parameter could be uniquely determined from the constituent measurements W_{α_k} or W_{β_k}. A substantial amount of information, therefore, can be lost when independent measurements are reduced by a finite difference process. It is easy to verify, however, that no information about the expected value of V_k is ever lost when a joint measure of W_{α_k} and W_{β_k} is replaced by V_k.

VII. CONCLUSIONS

Coherence theory is used to analyze the statistical properties of ocean-acoustic intensity fluctuations measured after saturated multipath propagation. Previous analyses in this area have been implicitly limited to certain special cases for which the time-bandwidth product of the field measured from a given source is unity. In this paper, the intensity statistics of the saturated region are extended and generalized to be a function of the measurement time and temporal coherence of the received field. The resulting intensity distributions are therefore highly relevant to modern ocean-acoustic sonar and communication systems which employ time-bandwidth products that often exceed unity, or average over many independent samples.

More general expressions are also offered for the ''noise of multiple distant sources.'' Specifically, it is shown that a previous and well-known assumption used to describe this noise, that the *intensities* received simultaneously from the noise sources are mutually independent, is a special case of the more general assumption adopted in this paper that the *fields* received from the noise sources are independent.

The statistics of averages of independent intensity samples are then examined because such amalgamation is commonly employed in a variety of ocean-acoustic measuring systems. The statistics of intensity rate measures obtained by finite difference are also examined to address the issue of monitoring a moving source or scatterer.

A brief discussion of classical parameter resolution bounds and Fisher information for the various distributions encountered is provided. This is used to show that parameter resolution is highly dependent upon the way that intensity measurements are made. For example, the logarithmic measures commonly used in ocean acoustics, such as scattering strength, target strength and TL, must be derived from a corrected version of the sonar equation that accounts for an inherent bias dependent on the time-bandwidth product of the intensity average. This bias attains its maximum magnitude of 2.5 dB for an instantaneous sample and only vanishes in the deterministic limit of large time-bandwidth product. The logarithmic measures then have mean-square errors that approximate the Cramer–Rao lower bound with increasing accuracy for increasing time-bandwidth product.

Finally, a quantitative measure is given for the amount of information that can be lost by certain widely practiced procedures for reducing a set of measurements to a single mean statistic. Such reduction is often employed in ocean-acoustic processing but may be detrimental to subsequent parameter estimates.

APPENDIX A: EULER'S PSI FUNCTION AND RIEMANN'S ZETA FUNCTION

Euler's psi function is defined by[27]

$$\psi(\mu)=-C-\sum_{k=0}^{\infty}\left(\frac{1}{\mu+k}-\frac{1}{k+1}\right),\quad \text{for real}\ \ \mu, \tag{A1}$$

$$=-C+\sum_{k=1}^{\mu-1}\frac{1}{k},\quad \text{for integer}\ \ \mu>1, \tag{A2}$$

$$=-C,\quad \text{for}\ \ \mu=1,$$

where C is Euler's constant,

$$C=\lim_{\mu\to\infty}\left\{\sum_{k=1}^{\mu-1}\frac{1}{k}-\ln\mu\right\}=0.577215\ldots\ . \tag{A3}$$

Riemann's zeta function is defined by[27]

$$\zeta(\upsilon,\mu)=\sum_{k=0}^{\infty}\frac{1}{(\mu+k)^{\upsilon}},\qquad \text{for}\ \ \upsilon>1,\ \ \mu\neq 0,-1,-2,-3,\ldots\ . \tag{A4}$$

APPENDIX B: VARIANCE OF THE INSTANTANEOUS INTENSITY OF A SUPERPOSITION OF INDEPENDENT CCGR FIELDS

Following Sec. III A, the instantaneous intensity for a sum of independent CCGR fields can be written as

$$I(t)=\sum_{i=1}^{S}z_i(t)\sum_{j=1}^{S}z_j^*(t), \tag{B1}$$

with expectation value

$$\langle I\rangle=E\left[\sum_{i=1}^{S}z_i(t)\sum_{j=1}^{S}z_j^*(t)\right]=\sum_{i=1}^{S}E[|z_i(t)|^2], \tag{B2}$$

where the last equality follows from the independence of the fields. The squared intensity has expectation value

$$\begin{aligned}\langle I^2\rangle&=E\left[\sum_{i=1}^{S}z_i(t)\sum_{j=1}^{S}z_j^*(t)\sum_{k=1}^{S}z_k(t)\sum_{l=1}^{S}z_l^*(t)\right]\\&=E\left[\sum_{i=1}^{S}z_i(t)\sum_{j=1}^{S}z_j^*(t)\right]E\left[\sum_{k=1}^{S}z_k(t)\sum_{l=1}^{S}z_l^*(t)\right]\\&\quad+E\left[\sum_{k=1}^{S}z_k(t)\sum_{j=1}^{S}z_j^*(t)\right]E\left[\sum_{i=1}^{S}z_i(t)\sum_{l=1}^{S}z_l^*(t)\right],\end{aligned} \tag{B3}$$

where the second equality follows from CCGR moment factoring.[2] Using Eq. (B2), this can be written as

$$\langle I^2\rangle=2\left(\sum_{i=1}^{S}E[|z_i(t)|^2]\right)^2, \tag{B4}$$

so that the variance is the square of the sum of the expected intensities of the superposed fields

$$\langle I^2\rangle-\langle I\rangle^2=\left(\sum_{i=1}^{S}E[|z_i(t)|^2]\right)^2. \tag{B5}$$

APPENDIX C: LIMITING FORMS FOR THE PROBABILITY DISTRIBUTION OF THE SUM OF TWO INDEPENDENT GAMMA-DISTRIBUTED INTENSITY MEASUREMENTS

When the expectation values of the two independent intensity samples of Sec. IV are nearly equal $\overline{I_\alpha}\approx\overline{I_\beta}$, the asymptotic form $I_\nu(\eta)\approx(\eta/2)^\nu/\Gamma(\nu+1)$ for small arguments η may be substituted for the modified Bessel function in Eq. (29), assuming ν is not a negative integer. The resulting distribution for the sum $\kappa=W_\alpha+W_\beta$ is then

$$P(\kappa)\approx\frac{2\sqrt{\pi}}{\Gamma(\mu+\frac{1}{2})\Gamma(\mu)}\left(\frac{\mu^2}{4\overline{I_\alpha I_\beta}}\right)^{\mu}\kappa^{2\mu-1}\times\exp\left\{-\mu\kappa\frac{(\overline{I_\alpha}+\overline{I_\beta})}{2\overline{I_\alpha I_\beta}}\right\}. \tag{C1}$$

After applying Gauss' multiplication formula for gamma functions,

$$\Gamma(n\chi)=(2\pi)^{1/2(1-n)}n^{n\chi-1/2}\prod_{k=0}^{n-1}\Gamma\left(\chi+\frac{k}{n}\right), \tag{C2}$$

for the case $n=2$, and transforming the sum $\kappa=W_\alpha+W_\beta$ to the average $\kappa=(W_\alpha+W_\beta)/2$, it is readily verified that Eq. (C1) reduces to

$$P(\kappa)=\frac{(2\mu/\overline{I})^{2\mu}\kappa^{2\mu-1}\exp\{-2\mu\kappa/\overline{I}\}}{\Gamma(2\mu)}, \tag{C3}$$

when $\overline{I}=\overline{I_\alpha}=\overline{I_\beta}$. It is noteworthy that Eq. (C3) also corresponds to the probability distribution for unpolarized thermal light.[2]

Similarly, as η becomes sufficiently large, the modified Bessel function takes the form $I_\nu(\eta)\approx e^{\eta}/\sqrt{2\pi\eta}$. Applying this to Eq. (29) for the case $\overline{I_\alpha}\gg\overline{I_\beta}$ the asymptotic distribution is simply $\mathscr{G}(\overline{I_\alpha},\mu)$. These limits may also be verified by use of characteristic functions.

APPENDIX D: FISHER INFORMATION FOR INDEPENDENT GAMMA AND EXPONENTIAL-GAMMA-DISTRIBUTED MEASUREMENTS

As noted in Sec. VI, the Fisher information for the joint gamma-distributed measurements $\mathbf{W}$, obeying the conditional probability distribution

$$P(\mathbf{W}|\mathbf{a})=\prod_{k=1}^{N}\frac{(\mu_k/\overline{I_k}(\mathbf{a}))^{\mu_k}W_k^{\mu_k-1}\exp\{-\mu_kW_k/\overline{I_k}(\mathbf{a})\}}{\Gamma(\mu_k)} \tag{D1}$$

is the same as that for the exponential-gamma distributed measurements $\mathbf{L}$, obeying the conditional distribution

$$P(\mathbf{L}|\mathbf{a})=\prod_{k=1}^{N}\frac{\left(\dfrac{\mu_k}{\overline{I_{0_k}}(\mathbf{a})}\right)^{\mu_k}\exp\left\{-\mu_k\dfrac{\exp(L_k)}{\overline{I_{0_k}}(\mathbf{a})}+\mu_kL_k\right\}}{\Gamma(\mu_k)}. \tag{D2}$$

It is possible to demonstrate this by inspection of Eq. (47) with the additional knowledge that the matrices $\mathbf{J}(\langle\mathbf{W}\rangle)$ and $\mathbf{J}(\langle\mathbf{L}\rangle)$ have respective inverses that take the form of the asymptotic covariances of $\mathbf{W}$ and $\mathbf{L}$ in the Gaussian limit of large μ_k. For example, $\mathbf{J}(\langle\mathbf{W}\rangle)$ and $\mathbf{J}(\langle\mathbf{L}\rangle)$ are N by N diagonal matrices with respective elements

$$J_{ij}(\langle\mathbf{W}\rangle)=\mu_i/(\overline{I_i})^2\delta_{ij},\quad\text{and}\quad J_{ij}(\langle\mathbf{L}\rangle)=\mu_i\delta_{ij}.$$

The more straight forward approach prescribed by Eq. (41) is instead presented. For the Fisher information matrix given the measurements $\mathbf{W}$, substitution of the right-hand side of Eq. (D1) in to Eq. (41) yields

$$J_{ij}(\mathbf{a}) = -E\left[\frac{\partial}{\partial a_i}\sum_{k=1}^{N} -\mu_k \frac{\partial}{\partial a_j}\left(\ln \overline{I_k} + \frac{W_k}{\overline{I_k}}\right)\right]. \tag{D3}$$

Application of the derivative operator with respect to a_j leads to

$$J_{ij}(\mathbf{a}) = -E\left[\frac{\partial}{\partial a_i}\sum_{k=1}^{N} -\mu_k\left(\frac{1}{\overline{I_k}} - \frac{W_k}{(\overline{I_k})^2}\right)\frac{\partial \overline{I_k}}{\partial a_j}\right]. \tag{D4}$$

Taking the derivative with respect to a_i leads to

$$J_{ij}(\mathbf{a}) = E\left[\sum_{k=1}^{N} \frac{\mu_k}{(\overline{I_k})^2}\left\{\left(2\frac{W_k}{\overline{I_k}} - 1\right)\frac{\partial \overline{I_k}}{\partial a_i}\frac{\partial \overline{I_k}}{\partial a_j} + (\overline{I_k} - W_k)\frac{\partial^2 \overline{I_k}}{\partial a_i\, \partial a_j}\right\}\right]. \tag{D5}$$

Finally, application of the expectation value operator yields Eq. (42) which can be rewritten as

$$J_{ij}(\mathbf{a}) = \sum_{k=1}^{N} \mu_k \frac{\partial \ln \overline{I_k}(\mathbf{a})}{\partial a_i}\frac{\partial \ln \overline{I_k}(\mathbf{a})}{\partial a_j}. \tag{D6}$$

Here, $\overline{I_k}(\mathbf{a})$ can be replaced by $\overline{I_{0_k}}(\mathbf{a}) = \overline{I_k}(\mathbf{a})/I_{\mathrm{ref}}$ to reference the logarithmic measure to physical units without altering the Fisher information.

For the Fisher information matrix given the measurements $\mathbf{L}$, substitution of the right-hand side of Eq. (D2) in to Eq. (41) yields

$$J_{ij}(\mathbf{a}) = -E\left[\frac{\partial}{\partial a_i}\sum_{k=1}^{N} -\mu_k \frac{\partial}{\partial a_j}\left(\ln \overline{I_{0_k}} + \frac{\exp(L_k)}{\overline{I_{0_k}}}\right)\right]. \tag{D7}$$

Because W_k/I_{ref} equals $\exp(L_k)$ by definition, and I_{ref} does not depend on $\mathbf{a}$, Eq. (D7) is identical to Eq. (D3). Therefore, Eq. (D7) reduces to

$$J_{ij}(\mathbf{a}) = \sum_{k=1}^{N} \mu_k \frac{\partial \ln \overline{I_{0_k}}(\mathbf{a})}{\partial a_i}\frac{\partial \ln \overline{I_{0_k}}(\mathbf{a})}{\partial a_j}, \tag{D8}$$

which equals Eq. (D6) as expected.

It is noteworthy that Eq. (D6) can be obtained directly from the Gaussian form of $P(\mathbf{W}|\mathbf{a})$ or $P(\mathbf{L}|\mathbf{a})$ in the asymptotic limit of $\mu_k \gg 1$, as is consistent with the fact that both the intensity measurement $\mathbf{W}$ and intensity level measurement $\mathbf{L}$ contain the same information.

APPENDIX E: THE LOSS OF INFORMATION IN COMBINING NONIDENTICALLY DISTRIBUTED FIELDS OR INTENSITIES

It is well established that the Fisher information matrix for a joint set of independent measurements equals the sum of the Fisher information matrices for each individual measurement.[36,38–39] For example, given scalar parameter a, the conditional probability distribution for the joint measurements of S independent and instantaneous CCGR fields z_i is given by the product

$$\prod_{i=1}^{S} P_{z_i}(z_i|a). \tag{E1}$$

The corresponding Fisher information is then

$$J_{\mathrm{joint}}(a) = \sum_{i=1}^{S} \frac{1}{(\overline{I_i})^2}\left(\frac{\partial \overline{I_i}}{\partial a}\right)^2. \tag{E2}$$

Following Sec. VI C, the Fisher information for an instantaneous measurement of the sum z of the fields z_i is

$$J_{\mathrm{sum}}(a) = \frac{1}{(\sum_{i=1}^{S}\overline{I_i})^2}\left(\frac{\partial}{\partial a}\sum_{j=1}^{S}\overline{I_j}\right)^2, \tag{E3}$$

where this information is unchanged if the sum z is replaced by $z/\sqrt{S}$. As a consequence of the positive semidefiniteness of expected intensity and the presence of squared terms, the following relationship must hold true

$$\left\{\sum_{i=1}^{S}\sum_{j=1}^{i}(1-\delta_{ij})\left(\frac{\overline{I_j}}{\overline{I_i}}\frac{\partial \overline{I_i}}{\partial a} - \frac{\overline{I_i}}{\overline{I_j}}\frac{\partial \overline{I_j}}{\partial a}\right)^2\right\} + \left\{\sum_{k=1}^{S}\left(\frac{1}{\overline{I_k}}\frac{\partial \overline{I_k}}{\partial a}\right)^2\left(\sum_{l=1}^{S}\sum_{m=1}^{l}(1-\delta_{lm})2\overline{I_l I_m}\right)\right\} \geqslant 0, \tag{E4}$$

where δ_{ij} is the Kronecker delta. By straight forward algebraic manipulation of relation (E4) it can be shown that

$$J_{\mathrm{joint}}(a) \geqslant J_{\mathrm{sum}}(a). \tag{E5}$$

Therefore, the joint measurements of the S fields z_i contain more information than a single measurement of the sum z of these fields. When these fields are identically distributed, the joint measurements contain precisely S times the information of the sum. More generally, dividing the left-hand side of relation (E4) by

$$\left(\sum_{k=1}^{S}\overline{I_k}\right)^2, \tag{E6}$$

gives the information about parameter a that is lost by retaining the sum rather than the joint measurements.

Similarly, given scalar parameter a, the conditional probability distribution for the joint measurement of D independent gamma-distributed intensity measurements W_i is given by the product

$$\prod_{i=1}^{D} P_{W_i}(W_i|a). \tag{E7}$$

The corresponding Fisher information is then

$$\mathscr{I}_{\mathrm{joint}}(a) = \sum_{i=1}^{D}\frac{\mu}{\langle W_i\rangle^2}\left(\frac{\partial\langle W_i\rangle}{\partial a}\right)^2, \tag{E8}$$

when the time-bandwidth product of each W_i equals μ. Following Sec. VI D, the Fisher information for the average κ of these D gamma-distributed intensities W_i is

$$\mathscr{I}_{\mathrm{average}}(a) = \frac{\mu}{\sum_{j=1}^{D}\langle W_j\rangle^2}\left(\frac{\partial}{\partial a}\sum_{i=1}^{D}\langle W_i\rangle\right)^2. \tag{E9}$$

The following relationship must hold true:

$$\left\{\sum_{i=1}^{D}\sum_{j=1}^{i}(1-\delta_{ij})\left(\frac{\langle W_j\rangle}{\langle W_i\rangle}\frac{\partial\langle W_i\rangle}{\partial a} - \frac{\langle W_i\rangle}{\langle W_j\rangle}\frac{\partial\langle W_j\rangle}{\partial a}\right)^2\right\} \geqslant 0 \tag{E10}$$

because the left-hand side of the equation must be positive semidefinite due to the squared terms. By straight forward algebraic manipulation of relation (E10) it can be shown that

$$\mathscr{I}_{\text{joint}}(a) \geqslant \mathscr{I}_{\text{average}}(a), \tag{E11}$$

where equality holds when the W_i are identically distributed. The joint measurements of the D intensities W_i, therefore, contain more information than a single measurement of average intensity κ unless the intensities are identically distributed, in which case the joint measurements and average have equal information. More generally, the left-hand side of relation (E10) divided by

$$\sum_{k=1}^{D} \frac{\langle W_k \rangle^2}{\mu}, \tag{E12}$$

gives the information about parameter a that is lost by keeping the average rather than the joint measurements.

Finally, by letting $\langle W_i \rangle = I_i$, $D=S$, $\mu=1$, and then comparing Eq. (E9) with Eq. (E3), one can see that information is always gained by measuring the instantaneous intensities of the S independent fields separately and then averaging them, rather than by first measuring the intensity of the superposed fields and then dividing by S. Although this gain of information can be substantial, it requires the ability to measure the intensity of each independent field z_i separately, which is not always possible for the reasons given in Sec. III B.

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